SYLLABI-BOOK MAPPING TABLE

Analysis

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BLOCK II: CONTINUITY AND POWER SERIES **Unit-5:** Complete Metric spaces: Introduction-completeness-Baire's Category theorem. **Unit-6:** Continuity-Homeomorphism-Uniform continuity. **Unit-7:** Differentiability of a function-Derivability & Continuity-Algebra of derivatives. **Unit-8:** Rolle's theorem-Mean Valued Theorems on derivatives-Taylor's Theorem with remainder-Power series expansion.	**Unit-V:** Complete metric spaces **(Pages 73-90);** **Unit-VI:** Continuity **(Pages 91-110);** **Unit-VII:** Differentiable function **(Pages 111-120);** **Unit-VIII:** Power series **(Pages 121-150);**
BLOCK III: INTEGRAL FUNCTIONS AND CONTRACTION MAPPING THEOREM **Unit-9:** Riemann integration-definition-Daurboux's theorem-conditions of integrability-Integrabliity of continuous & monotonic functions. **Unit-10:** Integral functions-Properties of Integral functions-Continuity & derivability of integral functions-The First Mean Value Theorem and the Fundamental theorem of Calculus. **Unit-11:** Contraction mapping-Definition and Examples-Contraction mapping theorem-Applications.	**Unit-IX:** Riemann integration **(Pages 151-162);** **Unit-X:** Integral functions **(Pages 163-186);** **Unit-XI:** *Contraction mapping and its applications* **(Pages 187-194);**
BLOCK IV: CONNECTED AND COMPACT METRIC SPACES **Unit-12:** Connectedness: Introduction-Connectedness definition and examples-Connected subsets of $\mathbb{R}$-Connectedness and Continuity. **Unit-13:** Compactness: Introduction-Compact metric spaces-Continuous functions on compact metric spaces- Continuity of the inverse function-Uniform continuity. **Unit-14:** Sequence of functions and Series of functions-Point wise Convergent-Cauchy criterion for uniform convergence.	**Unit-XII:** Connectedness **(Pages 195-206);** **Unit-XIII:** Compactness **(Pages 207-230);** **Unit-XIV:** Sequence of functions and series of functions **(Pages 231-240);**

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INTRODUCTION

NOTES

Analysis is the branch of mathematics that deals with inequalities and limits. The present course deals with the most basic concepts in analysis. The goal of the course is to acquaint the reader with rigorous proofs in analysis and also to set a firm foundation for calculus of one variable.

Calculus has prepared you, the student, for using mathematics without telling you why what you learned is true. To use, or teach, mathematics effectively, you cannot simply know what is true, you must know why it is true. This course shows you why calculus is true. It is here to give you a good understanding of the concept of a limit, the derivative, and the integral. Let us use an analogy.

We start with a discussion of the real number system, most importantly its completeness property, which is the basis for all that comes after. We then discuss the simplest form of a limit, the limit of a sequence. Afterwards, we study functions of one variable, continuity, and the derivative. Next, we define the Riemann integral and prove the fundamental theorem of calculus. We discuss sequences of functions and the interchange of limits. Finally, we give an introduction to metric spaces.

The term real analysis is a little bit of a misnomer. I prefer to use simply analysis. The other type of analysis, complex analysis, really builds up on the present material, rather than being distinct. Furthermore, a more advanced course on real analysis would talk about complex numbers often. I suspect the nomenclature is historical baggage.

BLOCK- I
SETS, FUNCTIONS AND METRIC SPACES

UNIT-I SETS AND FUNCTION

Structure

1.0 INTRODUCTION

In this chapter we introduce concepts which we need in the sequel.

1.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by sets and functions.
- Discuss intervals in ℝ.
- Describe countable and uncountable sets.

1.2 SETS AND FUNCTIONS

The concepts of sets and functions are indispensable to almost all branches of pure mathematics. The usual material of elementary set theory is so current that we take it for granted. We freely use the following notations of set theory.

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(i) A is a subset of B written as $A \subseteq B$.
(ii) Union of two sets A and B written as $A \cup B$.
(iii) Intersection of two sets A and B written as $A \cap B$.
(iv) Complement of subset of A of X written as A^c.
(v) Difference of two sets A and B written as $A - B$.
(vi) Cartesian product of two sets A and B written as $A \times B$.
(vii) A function f a set A to a set B written as $f: A \to B$.
(viii) The empty set which contains no elements is denoted by Φ.

Certain letters are reserved to denote particular sets which occur often in our discussion. They are

$\mathbb{N}$, the set of all natural numbers.
$\mathbb{Z}$, the set of all integers.
$\mathbb{Q}$, the set of all rational numbers.
$\mathbb{Q}^+$, the set of all positive rational numbers.
$\mathbb{R}$, the set of all real numbers.
$\mathbb{R}^n$, the set of all ordered n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers.
$\mathbb{C}$, the set of all complex numbers.
$\mathbb{C}^n$, the set of all ordered n-tuples $(x_1, x_2, \dots, x_n)$ of complex numbers.

The concept of union and intersection can be extended to any collection of sets. Let I be a nonempty set. For each $i \epsilon I$, let A_i be a set. Then we say that $\{A_i: i \epsilon I\}$ is a family of sets indexed by the set I.

We define $\bigcup_{i \epsilon I} A_i = \{x: x \epsilon A_i \, for \; atleast \; one \; i \epsilon I\}$ and
$$\bigcap_{i \epsilon I} A_i = \{x: x \epsilon A_i \, for \; all \; i \epsilon I\}.$$

Example 1. For each $i \epsilon \mathbb{N}$. Let $A_i = \{i, i+1, \dots, i+n, \dots\}$. Therefore, $A_1 = \{1,2, \dots\}; A_2 = \{2,3, \dots\}; \dots$. Then $\{A_i: i \epsilon \mathbb{N}\}$ is a family of sets indexed by $\mathbb{N}$. Here $\bigcup_{i \epsilon \mathbb{N}} A_i = \{1,2, \dots, n, \dots\} = \mathbb{N}$ and $\bigcap_{i \epsilon \mathbb{N}} A_i = \Phi$.

Note 1. $\bigcup_{i \epsilon \mathbb{N}} A_i$ is also written as $\bigcup_{i \epsilon \mathbb{N}}^{\infty} A_i$ and $\bigcap_{i \epsilon \mathbb{N}} A_i$ as $\bigcap_{i \epsilon \mathbb{N}}^{\infty} A_i$.

Note 2. The distributive laws for union and intersection and De Morgan's laws for finite number of sets can be generalized to any collection of sets as follows.

(i) $(\bigcup_{i \epsilon \mathbb{N}} A_i)^c = \bigcap_{i \epsilon \mathbb{N}} A_i^c$.
(ii) $(\bigcap_{i \epsilon \mathbb{N}} A_i)^c = \bigcup_{i \epsilon \mathbb{N}} A_i^c$.
(iii) $A \cap (\bigcup_{i \epsilon \mathbb{N}} B_i) = \bigcup_{i \epsilon \mathbb{N}} (A \cap B_i)$.
(iv) $A \cup (\bigcap_{i \epsilon \mathbb{N}} B_i) = \bigcap_{i \epsilon \mathbb{N}} (A \cup B_i)$.

1.2.1 INTERVALS IN $\mathbb{R}$

Let $a, b \epsilon \mathbb{R}$ and $a < b$. Then

(i) $(a, b) = \{x / x \epsilon \mathbb{R} \; and \; a < x < b\}$ is a called the **open interval** with a and b as end points.

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(ii) $[a, b] = \{x/x\epsilon\ \mathbb{R}\ and\ a \leq x \leq b\}$ is a called the **closed interval** with a and b as end points.

(iii) $(a, b] = \{x/x\epsilon\ \mathbb{R}\ and\ a < x \leq b\}$ is a called the **open-closed interval** with a and b as end points.

(iv) $[a, b) = \{x/x\epsilon\ \mathbb{R}\ and\ a \leq x < b\}$ is a called the **closed-open interval** with a and b as end points.

(v) $[a, \infty) = \{x/x\epsilon\ \mathbb{R}\ and\ x \geq a\}$.

(vi) $(a, \infty) = \{x/x\epsilon\ \mathbb{R}\ and\ x > a\}$.

(vii) $(-\infty, a] = \{x/x\epsilon\ \mathbb{R}\ and\ x \leq a\}$.

(viii) $(-\infty, a) = \{x/x\epsilon\ \mathbb{R}\ and\ x < a\}$.

(ix) $(-\infty, \infty) = \mathbb{R}$.

Any subset of $\mathbb{R}$ which is one of the above forms is called an **interval.** Any interval of the form (i),(ii),(iii) or (iv) is called a **finite interval or bounded interval** and any interval of the form (v), (vi), (vii), (viii) or (ix) is called an **infinite interval or an unbounded interval.**

The singleton set $\{a\}$ is considered to be a degenerated closed interval $[a, a]$.

1.3 COUNTABLE SETS

In this section we introduce the notation of countability and uncountability of a set. If a set A is finite then we can actually count the number of elements in this set. In other words we can label the elements of A by using the natural number $1,2, \dots, n$ for some n and the number of elements in this set A in n.

In this case there exists a bijection from A to B are two finite sets having the same number of elements, then there exists a bijection from A to B.

Definition. Two sets A and B are said to be **equivalent** if there exists a bijection f from A to B.

Note. From what we have seen above, two finite sets A and $B$are equivalent iff they have the same number of elements. Hence a finite set cannot be equivalent to a proper subset of itself. However, the infinite set can be equivalent to a proper subset as seen in the following examples.

Example 1. Let $A = \mathbb{N}$ and $B = \{2,4,6, \dots, 2n, \dots\}$.

Then $f: A \to B$ defined by $f(n) = 2n$ is a bijection. Hence A is equivalent to B even though A has actually 'more' elements than B.

Definition. A set A is set to be **countably infinite** if A is equivalent to the set of natural numbers $\mathbb{N}$.

A is said to be **countable** if it is finite or countably infinite.

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Note. Let A be a countably infinite set. Then there is a bijection f from $\mathbb{N}$ to A. Let $f(1) = a_1, f(2) = a_2, \ldots, f(n) = a_n, \ldots$. Then

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}.$$

Thus all the elements of A can be labeled by using the elements of $\mathbb{N}$.

Example 1. $\{2,4,6, \ldots, 2n, \ldots\}$ is a countable set.
Example 2. $\mathbb{Z}$ is countable (refer example 2).
Theorem 1. A subset of a countable set is countable.
Proof. Let A be a countable set and let $B \subseteq A$.

If A or B is finite, then obviously B is countable.

Hence let A and B be both infinite.

Since A is countably infinite, we can write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let a_{n_1} be the first element in A such that $a_{n_1} \epsilon B$. Let a_{n_2} be the second element in A which follows a_{n_1} such that $a_{n_1} \epsilon B$.

Proceeding like this we get $A = \{a_{n_1}, a_{n_2}, \ldots\}$. Thus all the elements of B can be labelled by using the elements of N. Hence B is countable.

Theorem 2. $\mathbb{Q}^+$ is countable.

Proof. Take all positive rational numbers whose numerator and denominator add up to 2. We have only one number namely $\frac{1}{1}$.

Next we take all positive rational numbers whose numerator and denominator add up to 3. We have $\frac{1}{2}$ and $\frac{2}{1}$.

Next we take all positive rational numbers whose numerator and denominator add up to 4. We have $\frac{3}{1}, \frac{2}{2}$ and $\frac{1}{3}$.

Proceeding like this, we can list all the positive rational numbers together from the beginning omitting those which are already listed.

Thus we obtain the set $\left\{1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, \ldots\right\}$. This set contains every positive rational number each occurring exactly once. Thus $\mathbb{Q}^+$ is countable.

Theorem 3. $\mathbb{Q}$ is countable.
Proof. We know that $\mathbb{Q}$ is countable. Let $\mathbb{Q}^+ = \{r_1, r_2, \ldots, r_n, \ldots\}$.

Therefore, $\mathbb{Q} = \{0, \pm r_1, \pm r_2, \ldots, \pm r_n, \ldots\}$.

Let $f: \mathbb{N} \to \mathbb{Q}$ be defined by $f(1) = 0, f(2n) = r_n$ and $f(2n+1) = -r_n$.

Clearly, f is a bijection and hence $\mathbb{Q}$ is countable.

Theorem 4. $\mathbb{N} \times \mathbb{N}$ is countable.
Proof. $\mathbb{N} \times \mathbb{N} = \{(a, b) | a, b \in \mathbb{N}\}$.

Take all ordered pairs (a, b) such that $a + b = 2$.

There is only one such pair namely (1,1).

Next take all ordered pairs (a, b) such that $a + b = 3$.

We have (1,2) and (2,1).

Next take all ordered pair (a, b) such that $a + b = 4$.

We have (3,1), (2,2) and (1,3).

Proceeding like this and listing all the ordered pairs together from the beginning, we get the set $\{(1,1), (1,2), (2,1), (3,1), (2,2), (1,3), \dots\}$. This set contains every ordered pair belonging to $\mathbb{N} \times \mathbb{N}$ exactly once.

Thus $\mathbb{N} \times \mathbb{N}$ is countable.

Theorem 5. If A and B are countable sets then $A \times B$ is also countable.
Proof. We assume that A and B are countably infinite.

Let $A = \{a_1, a_2, \dots, a_n, \dots\}; B = \{b_1, b_2, \dots, b_n, \dots\}$.
Now define $f: \mathbb{N} \times \mathbb{N} \to A \times B$ by $f(i, j) = (a_i, b_j)$.
We claim that f is a bijection.
Suppose $x = (p, q) \in \mathbb{N} \times \mathbb{N}$ and $y = (u, v) \in \mathbb{N} \times \mathbb{N}$.
Now,

$$
\begin{aligned}
f(x) = f(y) &\Longrightarrow (a_p, b_q) = (a_u, b_v) \\
&\Longrightarrow a_p = b_u, a_q = b_v \\
&\Longrightarrow p = v \text{ and } q = v \\
&\Longrightarrow x = y.
\end{aligned}
$$

$\therefore f$ is 1-1.
Now, suppose $(a_m, a_n) \in A \times B$.
Then $(m, n) \in \mathbb{N} \times \mathbb{N}$ and $f(m, n) = (a_m, a_n)$.
Therefore, f is onto. Hence f is a bijection.
Hence $A \times B$ is equivalent to $\mathbb{N} \times \mathbb{N}$ which is countable. (refer Theorem 4)
Hence $A \times B$ is countable.

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Theorem 6. Let A be a countably infinite set and f be a mapping of A onto a set B. Then B is countable.

Proof. Let A be a countably infinite set and $f: A \to B$ be an onto map.

Let $b \in B$. Since f is onto, there exist at least one pre-image for b. Choose one element $a \in A$ such that $f(a) = b$.

Now, define $g: B \to A$ by $g(b) = a$.

Clearly, g is $1 - 1$.

Therefore, B is equivalent to a subset of the countable set A.

Therefore, B is countable. (by theorem 1)

Theorem 7. Countable union of countable sets is countable.

Proof. Let $S = \{A_1, A_2, \dots, A_n, \dots\}$ be a countable family of countable sets.

Case (i) Let each A_i be countably infinite.

Let $A_1 = \{a_{11}, a_{12}, \dots, a_{1n}, \dots\}$

$A_2 = \{a_{21}, a_{22}, \dots, a_{2n}, \dots\}$

… … … …

… … … …

$A_n = \{a_{1n}, a_{1n}, \dots, a_{nn}, \dots\}$

… … … …

… … … …

Now we define a map $f: \mathbb{N} \times \mathbb{N} \to \bigcup A_n$ by $f(i, j) = a_{ij}$.

Clearly f is onto.

Also by theorem 4, $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Hence by theorem 6, $\bigcup A_n$ is countably infinite.

Case (ii) Let each A_i be countable.

For each i choose a set B_i such that B_i is a countably infinite set and $A_i \subseteq B_i$.

Now, $\bigcup A_i \subseteq \bigcup B_i$.

Now, $\bigcup B_i$ is countable (by case (i)).

Therefore, $\bigcup A_i$ is countable. (theorem 1).

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Problem 1. Any countable infinite set is equivalent to a proper subset of itself.

Solution. Let A be a countably infinite set.

Hence $A = \{a_1, a_2, \ldots, a_n, \ldots\}$.

Let $B = \{a_2, a_3, \ldots, a_n, \ldots\}$

Clearly B is a proper subset of A.

Define a map $f: A \to B$ by $f(a_n) = a_{n+1}$.

Clearly f is a bijection. Hence A is equivalent to B.

Problem 2. Any infinite set contains a countably infinite subset.

Solution. Let A be an infinite set.

Choose any element $a_1 \in A$.

Now, since A is infinite set, we can choose another element, $a_2 \in A - \{a_1\}$.

Now, suppose we have chosen $a_1, a_2, \ldots, a_n$ from A.

Since A is infinite, $A - \{a_1, a_2, \ldots, a_n\}$ is also infinite.

∴ We can choose a_{n+1} from $A - \{a_1, a_2, \ldots, a_n\}$.

Now, $B = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$ is countably infinite subset of A.

Problem 3. Any infinite set is equivalent to a proper subset of itself.

Solution. Let A be an infinite set.

By problem 2 above, A contains a countably infinite subset $B = \{a_2, a_3, \ldots, a_n, \ldots\}$.

Clearly $A = (A - B) \cup \{a_2, a_3, \ldots, a_n, \ldots\} = A - \{a_1\}$.

Clearly C is a proper subset of A.

Consider the function $f: A \to C$ defined by $f(x) = x$ if $x \in A - B$ and $f(a_n) = a_{n+1}$.

Obviously f is a bijection. Hence A is equivalent to C.

1.4 UNCOUNTABLE SETS

Definition. A set which is not countable is called **uncountable.** All the infinite sets we have considered in the previous section are countable.We shall now give an example of an uncountable set.

Theorem 8. (0,1] is uncountable.

Proof. Every real number in (0,1] can be written uniquely as a non-terminating decimal $0.a_1a_2 \ldots a_n \ldots$ where $0 \leq a_i \leq 9$ for each i

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subject to the following restriction that any terminating decimal $.a_1a_2 \dots a_n000 \dots$ is written as $.a_1a_2 \dots (a_n - 1)999 \dots$

For example $.54 = .53999 \dots.$

$1 = .999 \dots.$

Suppose (0,1] is countable.

Then the elements of (0,1] can be listed as $\{x_1, x_2, \dots x_n, \dots\}$ where

$x_1 = .a_{11}a_{12} \dots a_{1n} \dots$

$x_2 = .a_{21}a_{22} \dots a_{2n} \dots$

$\dots \dots \quad \dots \quad \dots$

$\dots \quad \dots \quad \dots \quad \dots$

$x_n = .a_{1n}a_{1n} \dots a_{nn} \dots$

$\dots \quad \dots \quad \dots \quad \dots$

$\dots \quad \dots \quad \dots \quad \dots$

Now, for each positive integer n choose an interger b_n such that $0 \leq b_n \leq 9$ and $b_n \neq 0$ and $b_n \neq a_{mn}$.

Let $y = .b_1b_2b_3 \dots$

Clearly $y \in (0,1]$.

Also y is different from each x_i at least in the i th place.

Hence $y \neq x_i$ for each i which is a contradiction.

Hence (0,1] is uncountable.

Corollary 1. Any subset A of $\mathbb{R}$ which contains (0,1] is uncountable.
Proof. Suppose A is uncountable.

Therefore, by theorem 1 any subset of A is countable. Hence we get (0,1] is countable which is a contradiction.Therefore, A is uncountable.

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Corollary 2. The set $\boldsymbol{S}$ of irrational numbers is uncountable.

Proof. Suppose **S** is countable.

We know $\mathbb{Q}$ is countable.

Therefore, $\boldsymbol{S} \cup \mathbb{Q} = \mathbb{R}$ is countable which is a contradiction. (by corollary 1). Therefore, **S** is uncountable.

1.5 INEQUALITIES OF HOLDER AND MINKOWSKI

Theorem 9 (Holder's Inequality). If $p > 1$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\sum_{i=1}^{n} |a_i b_i| \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots b_n$ are real numbers.

Proof. First we prove the inequality

$$x^{1/p} y^{1/q} \leq \frac{x}{p} + \frac{y}{q}, \text{where } x \geq 0 \text{ and } y \geq 0.$$

Now, let $x, y > 0$. Consider $f(t) = t^\lambda - \lambda t + \lambda - 1$ where $\lambda = \frac{1}{p}$ and $t \geq 0$.

Then $f'(t) = \lambda t^{\lambda-1} - \lambda = \lambda(t^{\lambda-1} - 1)$.

$\therefore f(1) = f'(1) = 0$.

Also $f'(t) > 0$ for $0 < t < 1$ and $f'(t) < 0$ for $t > 1$.

$\therefore f(t) \leq 0$ for all $t \geq 0$ and in particular $f\left(\frac{x}{y}\right) \leq 0$.

$\therefore f\left(\frac{x}{y}\right) = \left(\frac{x}{y}\right)^\lambda + \lambda - 1 \leq 0.$

$\therefore \left(\frac{x}{y}\right)^{1/p} - \frac{1}{p}\left(\frac{x}{y}\right) + \frac{1}{p} - 1 \leq 0.$

Multiplying by y we get $x^{1/p} y^{(1-1/p)} - \frac{x}{p} - \left(1 - \frac{1}{p}\right) y \leq 0.$

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$\therefore\; x^{1/p} y^{(1-1/p)} - \frac{x}{p} - \frac{y}{q} \le 0.$ (since $1 - \frac{1}{q} = \frac{1}{p}$).

$x^{1/p} y^{1/q} \le \frac{x}{p} + \frac{y}{q}.$

Now to prove Holder's inequality, we apply the above inequality to the numbers $x_j = \frac{|a_j|^p}{\sum_{i=1}^{n} |a_i|^p}$; $y_j = \frac{|b_j|^q}{\sum_{i=1}^{n} |b_i|^q}$ for each $j = 1,2, \dots, n$.

We get

$$\frac{|a_j||b_j|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}} \le \frac{x_j}{p} + \frac{y_j}{q}$$

for all $j = 1,2, \dots, n$.

Adding these n inequalities we get

$$\frac{\sum_{i=1}^{n} |a_i||b_i|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}} \le \sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) \qquad (1)$$

Now, $\sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) = \frac{1}{p} \sum_{j=1}^{n} x_j + \frac{1}{q} \sum_{j=1}^{n} y_j$

$= \frac{1}{p} + \frac{1}{q}$ (since $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j = 1$)

$= 1.$

Using this in (1) we get $\sum_{i=1}^{n} |a_i||b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q} \therefore$ $\sum_{i=1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$

Note. If we put $p = 2 = q$ in Holder's inequality we get the following inequality which is known as **Cauchy-Schwarz inequality.**

$$\sum_{i=1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |b_i|^2\right]^{1/2}$$

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Theorem 10. (Minkowski's Inequality)

If p≥ 1, $\left[\sum_{i=1}^{n}|a_i+b_i|^p\right]^{1/p} \le \left[\sum_{i=1}^{n}|a_i|^p\right]^{1/p}\left[\sum_{i=1}^{n}|b_i|^p\right]^{1/p}$, where $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots b_n$ are real numbers.

Proof. This inequality is trivial when $p = 1$. Let $p > 1$.

Clearly,

$$\left[\sum_{i=1}^{n}|a_i+b_i|^p\right]^{1/p} \le \left[\sum_{i=1}^{n}(|a_i|+|b_i|)^p\right]^{1/p} \qquad (1)$$

Now, $\sum_{i=1}^{n}(|a_i|+|b_i|)^p = \sum_{i=1}^{n}(|a_i|+|b_i|)^{p-1}(|a_i| + |b_i|)$

$$= \sum_{i=1}^{n}|a_i|(|a_i|+|b_i|)^{p-1} + |b_i|\sum_{i=1}^{n}(|a_i|+|b_i|)^{p-1}$$

$$\le \left[\sum_{i=1}^{n}|a_i|^p\right]^{1/p}\left[\sum_{i=1}^{n}(|a_i|+|b_i|)^{(p-1)q}\right]^{1/q} + \left[\sum_{i=1}^{n}|b_i|^p\right]^{1/p}\left[\sum_{i=1}^{n}(|a_i|+|b_i|)^{(p-1)q}\right]^{1/q}$$

Where $\frac{1}{p} + \frac{1}{q} = 1$. (using Holder's inequality).

Now, since $\frac{1}{p} + \frac{1}{q} = 1$. we have $p + q = pq$.

Hence $(p-1)q = p$.

∴ Dividing by $\left[\sum_{i=1}^{n}(|a_i|+|b_i|)^p\right]^{1/q}$ we get

$$\left[\sum_{i=1}^{n}(|a_i|+|b_i|)^p\right]^{1-1/q} \le \left[\sum_{i=1}^{n}|a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n}|b_i|^p\right]^{1/p}$$

$$\therefore \left[\sum_{i=1}^{n}(|a_i|+|b_i|)^p\right]^{1/p} \le \left[\sum_{i=1}^{n}|a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n}|b_i|^p\right]^{1/p} \qquad (2)$$

From (1) and (2) we get the required inequality.

NOTES

CHECK YOUR PROGRESS

1. Show that $\mathbb{N}$ is equivalent to $\mathbb{Z}$.
2. Prove that the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$ is countable.
3. Show that $\mathbb{R}$ is uncountable.

1.6 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. The function $f: \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} \ if\ n\ is\ even \\ \frac{1-n}{2} \ if\ n\ is\ odd \end{cases}$$

is bijection. Hence $\mathbb{N}$ is equivalent $\mathbb{Z}$.

2. Let $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$. The function $f: \mathbb{N} \to A$ defined by $f(n) = \frac{n}{n+1}$ is a bijection. Hence A is countable.

3. The results follows directly by taking $A = \mathbb{R}$ in theorem 8.

1.7 SUMMARY

1. A is a subset of B written as $A \subseteq B$. Union of two sets A and B written as $A \cup B$. Intersection of two sets A and B written as $A \cap B$. Complement of subset of A of X written as A^c. Difference of two sets A and B written as $A - B$. Cartesian product of two sets A and B written as $A \times B$. A function f a set A to a set B written as $f: A \to B$. The empty set which contains no elements is denoted by Φ.
2. Two sets A and B are said to be **equivalent** if there exists a bijection f from A to B.
3. The concept of union and intersection can be extended to any collection of sets.
4. Any subset of $\mathbb{R}$ which is one of the above forms is called an **interval**.
5. Any interval of the form $(a,b), [a,b], [a,b), (a,b]$ is called a **finite interval or bounded interval** and any interval of the form $[a,\infty), (a,\infty), (-\infty,a]$, $(-\infty,a)$, $(-\infty,\infty)$ is called an **infinite interval or an unbounded interval.**
6. The singleton set $\{a\}$ is considered to be a degenerated closed interval $[a,a]$.
7. A set A is set to be **countably infinite** if A is equivalent to the set of natural numbers $\mathbb{N}$.
8. A is said to be **countable** if it is finite or countably infinite.

9. A set which is not countable is called **uncountable**. All the infinite sets we have considered in the previous section are countable.We shall now give an example of an uncountable set.

1.8 KEYWORDS

1. Interval: Any subset of $\mathbb{R}$ which is one of the above forms is called an interval.
2. **Open Interval:** $(a, b) = \{x/x\epsilon\ \mathbb{R}\ and\ a < x < b\}$.
3. **Closed Interval:** $[a, b] = \{x/x\epsilon\ \mathbb{R}\ and\ a \le x \le b\}$.
4. **Open-Closed Interval:** $(a, b] = \{x/x\epsilon\ \mathbb{R}\ and\ a < x \le b\}$.
5. **Closed-Open Interval:** $[a, b) = \{x/x\epsilon\ \mathbb{R}\ and\ a \le x < b\}$.
6. **Finite interval or bounded interval:** Any interval of the form $(a, b), [a, b], [a, b), (a, b]$ is called a finite interval or bounded interval.
7. **Infinite interval or an unbounded interval:**Any interval of the form $[a, \infty), (a, \infty), (-\infty, a]$, $(-\infty, a)$, $(-\infty, \infty)$ is called an infinite interval or an unbounded interval.
8. **Countably Infinite :** a set A is set to be countably infinite if A is equivalent to the set of natural numbers $\mathbb{N}$.
10. **Countable:** A is said to be countable if it is finite or countably infinite.
11. **Uncountable:** A set which is not countable is called uncountable.
12. **Equivalent:** Two sets A and B are said to be equivalent if there exists a bijection f from A to B.

1.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Let $A = \{1,2,3, \dots, n, \dots\}$ and $B = \{1,4,9, \dots, n^2, \dots\}$. Show that A and B are equivalent.
2. Show that $\mathbb{N}$ and $A = \{101, 102,103, \dots\}$ are equivalent.
3. Show that for any two sets A and B, the set $A \times B$ is equivalent to the set $B \times A$.
4. Prove that the set of all even integers is countably infinite.
5. Prove that the set of all points (x, y) in the Euclidean plane with rational coefficient is countable.
6. Prove that $\mathbb{C}$ is uncountable.
7. Prove that the set of all irrational numbers lying in the interval (0,1] is uncountable.

NOTES

1.10 FURTHER READINGS

1. Arumugam & Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2010.
2. Richard R. Goldberg, Methods of Real Analysis, Oxford & IBH Publishing Company, New Delhi.
3. D. Somasundaram & B. Choudhary, A first course in Mathematical Analysis, Narosa Publishing House, Chennai.
4. M.K. Singhal & Asha Rani Singhal, A First Course in Real Analysis, R. Chand & Co., June 1997 Edition.
5. Shanthi Narayan, A course of Mathematical Analysis, S. Chand & Co., 1995.

UNIT-2 METRIC SPACES

NOTES

Structure

2.0 INTRODUCTION

The concept of convergence of sequences of real numbers depends on the absolute value of the difference between any two real numbers. We observe that this absolute is nothing but the distance between the two numbers when they are considered as points on the real lin. For the study of the concepts like continuity and convergence the algebraic properties of $\mathbb{R}$ are irrelevant. This situation necessitates the study of sets in which a reasonable notation of distance is defined. A set equipped with a reasonable concept of distance is called a *metric space*. In this chapter we develop in a systematic manner the main facts about metric spaces.

2.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by metric spaces.
- Discuss limits in metric spaces.
- Describe continuous functions on metric spaces.

NOTES

2.2 METRIC SPACES

Definition. A **metric space** is a non-empty set M together with a function $d: M \times M \to \mathbb{R}$

Satisfying the following conditions.

(i) $d(x, y) \geq 0$ for all x, y in M

(ii) $d(x, y) = 0$ *if and only if* $x = y$

(iii) $d(x, y) = d(x, y)$ for all x, y in M

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y and z in M.

d is called a **metric or distance function** and $d(x, y)$ is called the distance between x and y.

Note. The metric space M with metric d is denoted by (M, d) or simply by M. In the previous definition (i) and (ii) are known as the non-negative property, (iii) as the symmetry and (iv) as the triangle inequality of the metric. we shall give below many examples of metric spaces.

Example 1. The function d defined by $d(x, y) = |x - y|$ is a metric for the set $\mathbb{R}$ of real numbers. With this distance function, $\mathbb{R}$ is a metric space denoted by $(\mathbb{R}, d)$. This metric d is called the **usual metric** for $\mathbb{R}$.

Proof. (i) $d(x, y) = |x - y|$ is a non-negative real number and $d(x, y) = 0$ iff $|x - y| = 0$. This implies and implied by $x = y$.

(ii) $d(x, y) = |x - y| = |-(y - x)| = d(y, x)$.

(iii) $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y|$.

That is $d(x, y) \leq d(x, z) + d(z, y)$.

Example 2. If $x = (x_1, x_2), y = (y_1, y_2)$ are any two points in $\mathbb{R}^2$, we can define three metrics d_1, d_2 and d_3 from $\mathbb{R}^2 \times \mathbb{R}^2$ into $\mathbb{R}$ as follows:

$$d_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \quad (1)$$

$$d_2(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad (2)$$

$$d_3(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \quad (3)$$

We shall verify that (1), (2) and (3) satisfy all the requirements of a metric.

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Let $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$ be any three points of $\mathbb{R}^2$. For providing (1) to be a metric, we as follows:

(i) Since $(x_1 - y_1)^2$ and $(x_2 - y_2)^2$ are non-negative real numbers, we see that (1) is non-negative. Hence, we see that $d(x, y) > 0$.

(ii) The $x = y$ implies and is implied by $x_1 = y_1$ and $x_2 = y_2$ so that $d(x, y) = 0$. Hence $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = d(y, x)$

Hence, we have $d(x, y) = d(y, x)$.

(i) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(x_1 - z_1 + z_1 - y_1)^2 + (x_2 - z_2 + z_2 - y_2)^2}$

If $a_1 = (x_1 - z_1), b_1 = (z_1 - y_1), a_2 = (x_2 - z_2)$ and $b_2 = (z_2 - y_2)$, we have $d(x, y) = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$.

By applying Minkowski's inequality, we get

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \leq \sqrt{(a_1{}^2 + a_2{}^2)} + \sqrt{(b_1{}^2 + b_2{}^2)}.$$

Substituting for a_1, a_2, b_1 and b_2, we get $d(x, y) \leq d(x, z) + d(z, y)$. Which proves the triangle inequality.

In the case of (2), we proceed as follows:

(i) Since $|x_1 - y_1| > 0$ and $|x_2 - y_2| > 0$, it follows that $d_2(x, y) = |x_1 - y_1| + |x_2 - y_2| > 0$.

(ii) The $x = y$ implies and is implied by $(x_1, x_2) = (y_1, y_2)$ so that $x_1 = y_1$ and $x_2 = y_2$. Hence $|x_1 - y_1| = 0$. Hence $|x_2 - y_2| = 0$. From this, we get $|x_1 - y_1| + |x_2 - y_2| = 0$ which proves that $d_2(x, y) = 0$ if and only if $x = y$.

(iii) $d_2(x, y) = |x_1 - y_1| + |x_2 - y_2|$
$d_2(x, y) = |y_1 - x_1| + |y_2 - x_2| = d_2(y, x)$.

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(iv) $d(x, y) = |x_1 - y_1| + |x_2 - y_2| = |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|$.

By using the property of the absolute value function,

$$d(x, y) \leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$$

$$= |x_1 - z_1| + |x_2 - z_2| + |z_1 - y_1| + |z_2 - y_2|$$

$$= d(x, z) + d(z, y).$$

Hence, we have $d(x, y) \leq d(x, z) + d(z, y)$.

To verify (3) to be a metric, first note that by max $\{|x_1 - y_1|, |x_2 - y_2|\}$, we mean the greater of the two numbers $|x_1 - y_1|$ and $|x_2 - y_2|$.

(i) Since $|x_1 - y_1|$ and $|x_2 - y_2|$ are non-negative numbers,

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} > 0$$

so that $d_3(x, y) > 0$.

(ii) $x = y$ implies and is implied by $x_1 = y_1$ and $x_2 = y_2$ so that $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. From this, we get max $\{|x_1 - y_1|, |x_2 - y_2|\} = 0$ if and only if $x = y$.

This proves that $d_3(x, y) = 0$ iff $x = y$.

(iii) $d_3(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|y_1 - x_1|, |y_2 - x_2|\} = d_3(y, x)$.

(iv) To verify the triangle inequality,

$d_3(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|x_1 - z_1 + z_1 - y_1|, |x_2 - z_2 + z_2 - y_2|\}$

If $x_1 - z_1 = a_1,\ x_2 - z_2 = a_2, z_1 - y_1 = b_1, z_2 - y_2 = b_2,$ then

$$d(x, y) = \max\{|a_1 + b_1|, |a_2 + b_2|\}.$$

Since the argument is similar to the contrary case, let us assume that $|a_1| < |a_2|, |b_1| < |b_2|$ and also

$$\max\{|a_1 + b_1|, |a_2 + b_2|\} = |a_2 + b_2|.$$

Now, $max\ \{|a_1|, |a_2|\} = |a_2|$ and $max\ \{|b_1|, |b_2|\} = |b_2|$.

Hence, $|a_1 + b_1| \leq |a_1| + |b_1| < |a_2| + |b_2|$

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So $\max\{|a_1 + b_1|, |a_2+b_2|\} = |a_2|+|b_2| \leq max\ \{|a_1|, |a_2|\} + max\ \{|b_1|, |b_2|\}$

Therefore, $d_3(x, y) \leq d_3(x, z) + d_3(z, y)$ which proves the triangle inequality.

Example 3. On any non-empty set M we define d as follows

$$d(x, y) = \begin{cases} 0\ if\ x = y \\ 1\ if\ x \neq y. \end{cases}$$

Then d is a metric on M. This is called the **discrete metric** on M.

Proof. Clearly, $d(x, y) \geq 0$ and $d(x, y) \Leftrightarrow x = y$.

Also $d(x, y) = d(y, x) = \begin{cases} 0\ if\ x = y \\ 1\ if\ x \neq y. \end{cases}$

$\therefore d(x, y) = d(y, x)$ for all $x, y \in M$.

Now let $x, y, z \in M$.

Case (i) $x = z$

Then $d(x, z) = 0$.

Also, $d(x, y) + d(y, z) \geq 0$.

$\therefore d(x, z) \leq d(x, y) + d(y, z)$.

Case (ii) $x \neq z$

Then $d(x, z) = 1$.

Also, since x, z are distinct, y can not be equal to both x and z.

Hence either $y \neq x$ or $y \neq z$.

$\therefore d(x, y) + d(y, z) \geq 1$.

$\therefore d(x, z) \leq d(x, y) + d(y, z)$.

Thus $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Hence d is a metric M.

Example 4. In $\mathbb{R}^n$ we define

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$$

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Where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. Then d is a metric on $\mathbb{R}^n$. This is called the **usual metric** on $\mathbb{R}^n$.

Proof. $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2} \geq 0.$

$d(x, y) = 0 \Leftrightarrow [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2} = 0.$

$\Leftrightarrow \sum_{i=1}^{n}(x_i - y_i)^2 = 0$ for all $i = 1,2, \dots, n$.

$\Leftrightarrow x_i = y_i$ for all $i = 1,2, \dots, n$.

$\Leftrightarrow (x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$.

$\Leftrightarrow x = y.$

Also, $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$

$= [\sum_{i=1}^{n}(y_i - x_i)^2]^{1/2}$

$= d(y, x).$

To prove the triangle inequality, take

$a_i = x_i - y_i$; $b_i = y_i - z_i$ and $p = 2$ in Minkowski's inequality we get,

$$\left[\sum_{i=1}^{n}(x_i - z_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n}(y_i - z_i)^2\right]^{1/2}$$

i.e.,$d(x, z) \leq d(x, y) + d(y, z)$.

$\therefore$ d is metric on $\mathbb{R}^n$.

Note. $\mathbb{R}^n$ with usual metric is called the **n-dimensional Euclidean space.**

Example 5. Consider $\mathbb{R}^n$. Let $p > 1$. We define $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^p]^{1/p}$

Where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. Then d is a metric on $\mathbb{R}^n$.

The proof is similar to that of example 4.

Example 6. Consider $\mathbb{R}^n$. Let $p \geq 1$. Let l_p denote the set of all sequence (x_n) such that $\sum_1^\infty |x|^p$ is convergent. Define $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^p]^{1/p}$

Where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.

Then d is a metric on $l_{p.}$

Proof. Let $a, b \in l_{p.}$

First we prove $d(a, b)$ is a real number.

By Minkowshi's inequality we have

$[\sum_{i=1}^{n} |a_i + b_i|^p]^{1/p} \leq [\sum_{i=1}^{n} |a_i|^p]^{1/p} + [\sum_{i=1}^{n} |b_i|^p]^{1/p}$.....(1)

Since $a, b \in l_{p.}$ the right hand side of (1) has a finite limit as $n \to \infty$.

$\therefore [\sum_{i=1}^{\infty} |a_i + b_i|^p]^{1/p}$is a convergent series.

Similarly we can prove that $[\sum_{i=1}^{\infty} |a_i - b_i|^p]^{1/p}$ is also a convergent series and hence $d(a, b)$ is a real number.

Now, taking the limit as $n \to \infty$ in (1) we get

$[\sum_{i=1}^{\infty} |a_i + b_i|^p]^{1/p} \leq [\sum_{i=1}^{\infty} |a_i|^p]^{1/p} + [\sum_{i=1}^{\infty} |b_i|^p]^{1/p}$.....(2)

Obviously $d(x, y) \geq 0$,

$d(x, y) = 0$ iff $x = y$.

And $d(x, y) = d(y, x)$.

Now, let $x, y, z \in l_{p.}$ Taking, $a_i = x_i - y_i$; $b_i = y_i - z_i$ in (2) we get

$$\left[\sum_{i=1}^{\infty} (x_i - z_i)^p\right]^{1/p} \leq \left[\sum_{i=1}^{\infty} (x_i - y_i)^p\right]^{1/p} + \left[\sum_{i=1}^{\infty} (y_i - z_i)^p\right]^{1/p}$$

i.e.,$d(x, z) \leq d(x, y) + d(y, z)$.

$\therefore$ d is metric on l_p.

Example 7. Let M be the set of all sequence in $\mathbb{R}$. Let $x, y \in M$ and let $x = (x_n)$ and $y = (y_n)$.

Define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$$

Then d is a metric on M.

Proof. Let $x, y \in M$. First we prove that $d(x, y)$ is a real number ≥ 0.

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We have $\frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} \le \frac{1}{2^n}$ for all n.

Also, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series.

$\therefore \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$ is a convergent series. (by comparison test)

$\therefore d(x, y)$ is a real number and $d(x, y) \ge 0$.

Now,

$$d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} = 0.$$

$\Leftrightarrow |x_n - y_n| = 0$ for all n.

$\Leftrightarrow x_n = y_n$ for all n.

$\Leftrightarrow x = y$.

Also, $d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$

$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n(1+|y_n - x_n|)}$

$= d(y, x)$.

Now, let $x, y, z \in M$. Then

$\frac{|x_n - z_n|}{1+|x_n - z_n|} \le 1 - \frac{1}{1+|x_n - z_n|} \le 1 - \frac{1}{(1+|x_n - y_n|+|y_n - z_n|)}$

$= \frac{|x_n - y_n|+|y_n - z_n|}{(1+|x_n - y_n|+|y_n - z_n|)}$

$= \frac{|x_n - y_n|}{(1+|x_n - y_n|+|y_n - z_n|)} + \frac{|y_n - z_n|}{(1+|x_n - y_n|+|y_n - z_n|)}$

$\le \frac{|x_n - y_n|}{(1+|x_n - y_n|)} + \frac{|y_n - z_n|}{(1+|y_n - z_n|)}$

Multiplying both side of this inequality by $\frac{1}{2^n}$ and take the sum from $n = 1$ to ∞ we get $d(x, z) \le d(x, y) + d(y, z)$.

$\therefore d$ is metric on M.

Example 8. Let d_1 and d_2 be two metrics on M. Define $d(x, y) = d_1(x, y) + d_2(x, y)$. Prove that d is a metric space on M.

Solution. $d(x, y) = d_1(x, y) + d_2(x, y) \ge 0$.

$d(x, y) = 0 \Leftrightarrow d_1(x, y) + d_2(x, y) = 0.$

$\Leftrightarrow d_1(x, y) = 0$ and $d_2(x, y) = 0.$

$\Leftrightarrow x = y.$

Now, $d(x, y) = d_1(x, y) + d_2(x, y)$

$= d_1(y, x) + d_2(y, x)$

$= d(y, x).$

Let $x, y, z \in M$. Then we have

$d_1(x, z) \leq d_1(x, y) + d_1(y, z)$ and

$d_2(x, z) \leq d_2(x, y) + d_2(y, z).$

Adding, we get $d(x, z) \leq d(x, y) + d(y, z).$

$\therefore d$ is a metric on M.

Example 9. Determine whether $d(x, y)$ defined on $\mathbb{R}$ by $d(x, y) = (x - y)^2$ is a metric or not.

Solution. Let $x, y \in \mathbb{R}$.

$d(x, y) = (x - y)^2 \geq 0.$

$d(x, y) = (x - y)^2 = (y - x)^2$

$= d(y, x).$

But triangle inequality does not hold.

Take $x = -5, y = -4$ and $z = 4$

Then $d(x, y) = (-5 + 4)^2 = 1$

$d(y, z) = (-4 - 4)^2 = 64$

$d(x, z) = (4 + 5)^2 = 81.$

Here $d(x, z) > d(x, y) + d(y, z)$

Hence triangle inequality does not hold.

$\therefore d$ is not a metric on $\mathbb{R}$.

2.3 LIMITS OF FUNCTIONS IN METRIC SPACES

For defining limits of functions in metric spaces, we need the notation of cluster points in a set and so we explain it briefly.

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Definition. Let (M, d) be a metric space and E be a subset of M. $a \in M$ is called a cluster point or a **limit point** of E if for every $r > 0$, there exists a $b \in E$ distinct from a such that $d(a, b) < r$.

That is, a is a cluster point of E, if there are points of E distinct from a which are arbitrarily close to a. It must be noted that the cluster point may or may not belong to the set.

Example 1. The set of cluster points of $B = (0,1)$ in $\mathbb{R}$ is $[0,1]$.

No point outside $(0,1)$ can be a cluster point of $(0,1)$. 0 and 1 are cluster points of B. Since for every $\varepsilon > 0$, we can find a point of $(0,1)$ in $(0, \varepsilon)$ distinct from 0. Similarly for the point 1 and other points of $(0,1)$. Hence, the set of cluster points of $(0,1)$ in $\mathbb{R}$ is $[0,1]$.

Example 2. The set of cluster points of $(0,1)$ in $\mathbb{R}_D$ is empty.

No point of $\mathbb{R}$ can be a cluster point of $(0,1)$ in $\mathbb{R}_D$. Suppose if a is a cluster point of $(0,1)$ in $\mathbb{R}_D$, then for every $\varepsilon > 0$, there should exist a b distinct from a such that $d(a, b) < \varepsilon$ which is not possible since $d(a, b) = 1$ when $a \neq b$ in $\mathbb{R}$. Hence, the set of all cluster points of $(0,1)$ in $\mathbb{R}_D$ is empty.

We shall now introduce the concept of the limit of a function in metric spaces.

Let (M_1, d_1) and (M_2, d_2) be metric spaces and let $a \in M_1$. Let f be a function whose range is contained in M_2 and whose domain contains all $x \in M_1$ such that $d_1(a, x) < r$ for some $r > 0$ except possibly at $x = a$. We also assume that a is a cluster point of the domain of f. That is, we assume that for every $r > 0$, there is a point b in the domain of f distinct from a such that $d_1(a, b) < r$.

Definition. $f(x)$ is said to approach L where $L \in M_2$ as x approaches a, if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_2(f(x), L) < \varepsilon$ when $0 < d_1(x, a) < \delta$. We denote this by $\lim_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$.

The following theorem gives the algebraic properties of the limits of real valued functions on metric spaces.

Theorem 1. Let (M, d) be a metric space and let a be a point in M. Let f and g be real valued functions whose domains are subset of M and ranges are in $\mathbb{R}$ with the usual absolute value metric. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$ where L, N are in $\mathbb{R}$, then we have,

(i) $\lim_{x \to a}[f(x) + g(x)] = L + N$.

(ii) $\lim_{x \to a}[f(x) - g(x)] = L - N$.

(iii) $\lim_{x \to a}[f(x).g(x)] = L\,N$.

(iv) $\lim_{x \to a}[\frac{f(x)}{g(x)}] = \frac{L}{N}$.

Proof. Proof follows exactly on the same lines as the proof in Theorem 4 of unit 7, when we replace the absolute value function in the domain by the respective metric d. So we omit the details of the proof.

2.4 CONTINUOUS FUNCTIONS ON METRIC SPACES

As in the case of the generalization of the limits of sequences and functions in metric spaces, we shall define continuous functions in a metric space (M, d) by replacing the absolute value value in the definition of continuity in $\mathbb{R}$ by the metric and creating the analogues for an interval in $\mathbb{R}$ with the help of the metric.

*A real valued function defined on $\mathbb{R}$ is said to be **continuous** at $x = a$ if $lim_{x \to a} f(x) = f(a)$. Since the function is defined at $x = a$, this definition is equivalent to the following $\varepsilon - \delta$ formulation.*

The real valued function f is continuous at $a \in \mathbb{R}$ if and only if given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whereever $|x - a| < \delta$.

Definition. Let (M, d) be a metric space. If $a \in M$ and $r > 0$, then an **open sphere** of radius r about a denoted by $B(a; r)$ is defined to be the set of all points in M whose distance to a is less than r. That is $B(a; r) = \{x \in M: d(x, a) < r\}$. Since $a \in B(a:r), B(a:r)$ is non-empty.

Example 1. The open sphere $B(a; r)$ on the real line is the bounded open interval $(a - r, a + r)$ with mid point a and total length $2r$ and $B(0; 1)$ is the bounded open interval $(-1,1)$.

Example 2. In Euclidean 3-space, $B(0; 1)$ is the set of all points (x, y, z)such that $x^2 + y^2 + z^2 < 1$ which has motivated the above terminology since $x^2 + y^2 + z^2 < 1$ is the inside of the sphere.

Example 3. Let $M = \mathbb{R}_D$, the real line with the discrete metric. Let a be any point in $\mathbb{R}_D$. For $0 < r < 1$, we have $B(a; r) = a$ because the only point in $\mathbb{R}_D$ whose distance from a is less than 1 is a itself. But $B(a; r) = \mathbb{R}_D$ for $r > 1$.

Since the open spheres in metric spaces are analogues of open intervals on the real line, we shall give below the definition of

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convergent sequence and continuous function using the open spheres.

Definition. A sequence (x_n) **converges** to a if and only if given $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $x_n \in B(a; \varepsilon)$ for all $n \geq n_0$.

Theorem 2. Let M_1 and M_2 be metric spaces with metrics d_1 and d_2 and let f be a mapping of M_1 into M_2. Then f is continuous at $a \in M_1$ if and only if any one and hence all of the following three conditions hold.

(i) Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), f(a)) < \varepsilon$ whenever $d_1(x, a) < \delta$.

(ii) The inverse image of f of any open sphere $B(f(a); \varepsilon)$ about $f(a)$ contains an open sphere $B(a; \delta)$ about a.

(iii) Whenever (x_n) is a sequence of points in M_1 converging to a, then the sequence $(f(x_n))$ of points in M_2 converges to $f(a)$.

Proof. (i) is the reformulation of the definition of continuous function using the metric d_1 and d_2 in M_1 and M_2 in the place of absolute value function.

(ii) Let us assume that f is continuous. Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), f(a)) < \varepsilon$ whenever $d_1(x, a) < \delta$. From this we get $f(x) \in B(f(a); \varepsilon)$ which shows that $x \in f^{-1}(B(f(a); \varepsilon))$. Since we consider only the values of x which lie in $B(a; \delta)$, we get

$$B(a; \delta) \subset f^{-1}(B(f(a); \varepsilon)). \qquad \text{......(1)}$$

Hence if f is continuous, the inverse image of any open sphere $B(f(a); \varepsilon)$ about $f(a)$ contains an open sphere $B(a; \delta)$ about a.

Conversely if $B(a; \delta) \subset f^{-1}B(f(a); \varepsilon)$, then we have $f(B(a; \delta)) \subset B(f(a); \varepsilon)$. This implies that whenever $x \in B(a; \delta)$, $f(x) \in B(f(a); \varepsilon)$. That is $d_2(f(x), f(a)) < \varepsilon$ whenever $d_1(x, a) < \delta$.

(iii) Let f be a continuous at a and prove that if $x_n \to a$ as $n \to \infty$, then $f(x_n) \to f(a)$ as $n \to \infty$. Note that $f(x_n)$ will be defined for large values of n. To prove the assertion, we have to show that given $\varepsilon > 0$, there exists positive integer n_0 such that $f(x) \in B(f(a); \varepsilon)$ for all $n \geq n_0$. Since f is continuous at a, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x) \in B(f(a); \varepsilon)$ whenever $x \in B(a; \delta)$. Hence, $f(B(a; \delta)) \subset B(f(a); \varepsilon)$. ....(2)

Since $x_n \to a$ as $n \to \infty$, there exists positive integer n_0 such that $x \in B(a; \delta)$ for all $n \geq n_0$. ....(3)

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Form (2) and (3), we see that

$f(x) \in B(f(a);\varepsilon)$ for all $n \geq n_0$. Hence, $f(x_n) \to f(a)$ as $n \to \infty$.

Conversely, $x_n \to a$ implies $f(x_n) \to f(a)$ as $n \to \infty$ and prove that f is continuous at $x = a$. Assume the contrary. Then by (ii), the inverse image under f of $B = B(f(a);\varepsilon)$ contains no open sphere about a. In particular $f^{-1}(B)$ does not contain $B\left(a;\frac{1}{n}\right)$ for any positive integer n. Hence, for each positive integer, there is a point $x_n \in B\left(a;\frac{1}{n}\right)$ such that $f(x) \notin B(f(a);\varepsilon)$. Hence $d_1(x_n,a) < \frac{1}{n}$ but $d_2\big(f(x_n),f(a)\big) > \varepsilon$. This contradicts the fact that $f(x_n) \to f(x)$ as $n \to \infty$. This contradiction proves the result.

Note 1. (i) can also be put in the following equivalent form. For each open sphere $B(f(a);\varepsilon)$ centred at $f(a)$, there exists an open sphere $B(a;\delta)$ centred at a such that $f\big(B(a;\delta)\big) \subset B(f(a);\varepsilon)$.

Note 2. To verify that a given function between metric spaces is continuous, the sequential characterization of continuous functions given in (iii) is more useful. We shall apply (iii) to prove that the properties of continuous functions also.

Note 3. All the above discussion in relation to convergence of sequence and continuous functions given for metric spaces can be easily modified for real valued functions defined on metric spaces.

Theorem 3. Let (M_1,d_1) and (M_2,d_2) be metric spaces and let $f{:}M_1 \to M_2$, $g{:}M_2 \to M_3$. If f is continuous at $a \in M_1$ and g is continuous at $f(a) \in M_2$, then $g \circ f$ is continuous at a.

Proof. Let (x_n) be a sequence in M_1 such that $x_n \to a$ as $n \to \infty$. To prove the theorem, we have to show that $\lim_{n\to\infty} g[f(x_n)] = g[f(a)]$. Since f is continuous at a, we have $\lim_{n\to\infty} f(x_n) = f(a)$.

Let $y_n = f(x_n)$ and $y_n \to f(a)$ as $n \to \infty$ in (M_2,d_2). Since g is continuous $\lim_{n\to\infty} g(y_n) = g(f(a))$. substituting for y_n, we get $\lim_{n\to\infty} g(f(x_n)) = g(f(a))$. Hence $g \circ f\colon M_1 \to M_3$ is continuous .

To prove the result in $\mathbb{R}^2$, we can make use of any one of the three equivalent metrics in $\mathbb{R}^2$ given in Example. Without loss of generality, let us take the second metric in Exampl. Hence, we have $d_2\big(h(x_n,y_n),h(x,y)\big) = |f(x_n) - f(x)| + |g(y_n) - g(y)|$

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Using the hypothesis (1) and (2) in the above expression, we have

$d_2\big(h(x_n,y_n),h(x,y)\big) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ for all$n \geq n_0$. This shows that $h(x_n,y_n) \to h(x,y)$ as $n \to \infty$ in $\mathbb{R}^2$.

Example 4. Let $f: l^2 \to l^2$. Let $x = (x_1, x_2, \ldots) \in l^2$. Let $f(x)$ be defined as $f(x) = (0, x_1, x_2, \ldots)$, prove that f is continuous on l^2.

Let $s_n = (x_1{}^n, x_2{}^n, \ldots)$ and let s_n tend to $s = (x_1, x_2, \ldots)$ as $n \to \infty$ in l^2 metric. We shall show that $f(s_n) \to f(s)$as $n \to \infty$ in l^2.

$$d(s_n, s) = \sqrt{(x_1{}^n - x_1)^2 + (x_2{}^n - x_2)^2 + \cdots + (x_k{}^n - x_k)^2 + \cdots} \quad (1)$$

$$d(f(s_n), f(s)) = \sqrt{(x_1{}^n - x_1)^2 + (x_2{}^n - x_2)^2 + \cdots + (x_k{}^n - x_k)^2 + \cdots} \quad (2)$$

Since (1) and (2) are the same, $d(s_n, s) \to 0$ as $n \to \infty$ in l^2.
$d(f(s_n), f(s)) \to 0$ as $n \to \infty$ in l^2.
Hence f is a continuous function.

CHECK YOUR PROGRESS

1. If d is a metric on M, is d^2 a metric on M?
2. Is [0,1] is open ball in M?
3. Define continuous.

2.5 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Consider $d(x,y)$ defined on $\mathbb{R}$ by $d(x,y) = |x-y|$. We know that d is a metric on $\mathbb{R}$ (refer example 1). $d^2(x,y) = |x-y|^2 = (x-y)^2$. But d^2 is not a metric (refer example 9).

2. Let $X = [0,1]$ with absolute value metric $B\left(\frac{1}{2};\frac{1}{4}\right)$ is $B\left(\frac{1}{4};\frac{3}{4}\right)$ but $B\left(\frac{1}{4};\frac{1}{2}\right)$ is $\left[0,\frac{3}{4}\right)$, since points in $\mathbb{R}$ to the left of 0 are not in M.

3. A real valued function defined on $\mathbb{R}$ is said to be **continuous** at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

2.6 SUMMARY

1. A set equipped with a reasonable concept of distance is called a **metric space.**

2. d is called a **metric or distance function** and $d(x,y)$ is called the **distance** between x and y.

3. Let (M,d) be a metric space and E be a subset of M. $a \in M$ is called a cluster point or a **limit point** of E if for every $r > 0$, there exists a $b \in E$ distinct from a such that $d(a,b) < r$.

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4. $f(x)$ is said to **approach** L where $L \in M_2$ as x approaches a, if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_2(f(x), L) < \varepsilon$ when $0 < d_1(x, a) < \delta$. We denote this by $\lim_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$.
5. A real valued function defined on $\mathbb{R}$ is said to be **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.
6. A sequence (x_n) **converges** to a if and only if given $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $x_n \in B(a; \varepsilon)$ for all $n \geq n_0$.

2.7 KEYWORDS

1. **Metric Space:** A set equipped with a reasonable concept of distance is called a metric space.
2. **Metric or distance function** d is called a metric or distance function and $d(x, y)$ is called the distance between x and y.
3. **Usual metric:** A metric space denoted by $(\mathbb{R}, d)$ is defined by $d(x, y) = |x - y|$. This metric d is called the usual metric for $\mathbb{R}$.
4. **Discrete metric:** Any non-empty set M we define d as $d(x, y) = \begin{cases} 0 \text{ } if \text{ } x = y \\ 1 \text{ } if \text{ } x \neq y. \end{cases}$ Then d is a metric on M. This is called the discrete metric on M.
5. **n-dimensional Euclidean space:** $\mathbb{R}^n$ with usual metric is called the n-dimensional Euclidean space.

2.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Let (M, d) is a metric space. Define $d_1(x, y) = \min\{d(x, y), 1\}$. Prove that (M, d_1) is a bounded metric space.
2. Prove that in a metric space any subset of a bounded set is bounded.
3. In $\mathbb{R}$, with usual metric find $B(1,1)$.
4. In $\mathbb{R}^2$, with usual metric find $B((0,0), \frac{1}{2})$.
5. Determine $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \{1\}$ is open in $\mathbb{R}$ with usual metric.
6. Find the diameter of the following subset of $\mathbb{R}$ with usual metric.
   - i. $\{1,3,5,7,9\}$.
   - ii. $\mathbb{N}$.
   - iii. $\mathbb{Q}$.
   - iv. $[3,6] \cap [4,8]$.
7. Determine which of the following subsets of $\mathbb{R}$ are open in $\mathbb{R}$ with usual metric.
   - a. $\mathbb{R}$
   - b. $(1,2) \cup (3,4)$.
   - c. (a, ∞).

NOTES

2.9 FURTHER READINGS

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6.

UNIT-3 CONTINUOUS FUNCTIONS ON METRIC SPACES

NOTES

Structure

3.0 INTRODUCTION

Theorems about continuous real-valued functions on a closed bounded interval $[a, b]$ such as, "If f is continuous on $[a, b]$, then f takes on a maximum and minimum values,'' and "If f is continuous on $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$" are tools in the proof of the basic theorems in differential and integral calculus. We deduce these theorems as special cases of theorems about continuous functions on metric spaces. However, we first review the concept of continuity in its most elementary form.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by continuous functions in a point.
- Discuss reformulation.
- Describe bounded sets.

3.2 FUNCTION CONTINUOUS AT A POINT ON THE REAL LINE

Let a be a point in $\mathbb{R}$ and suppose f is a real-valued function whose domain contains all points of some open interval $(a-h, a+h)$ where $h > 0$ including a itself.

Definition. We say that the function f is continuous at $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$.

The definition really demands that two conditions be fulfilled in order that f be continuous at a. The first condition is that the $\lim_{x\to a} f(x)$ exists; the second is that this limit be equal to $f(a)$. In particular, if $f(a)$ is not defined, then f cannot be continuous at a. For example, the function f defined by

$$f(x) = \frac{\sin x}{x} \quad (x \in \mathbb{R}, x \neq 0)$$

is not defined at $x = 0$ and hence is not continuous at $x = 0$ even through $\lim_{x\to 0}(\sin x/x)$ exists (and is equal to 1).

However, the function g defined by

$$g(x) = \frac{\sin x}{x} \quad (x \neq 0),$$

$$g(0) = 1,$$

is continuous at $x = 0$ since $\lim_{x\to 0} g(x) = g(0)$.

It is often the case that a function f fails to be continuous at a point a because $\lim_{x\to a} f(x)$ does not exist; more frequently, indeed, than it fails because $f(a)$ is not defined or because $f(a)$ is not equal to $\lim_{x\to a} f(x)$. Consider, for example, the characteristic function χ of the rational numbers. That is,

$$\chi(x) = 1 \quad (x \in \mathbb{R}, x \text{ rational}),$$

$$\chi(x) = 0 \quad (x \in \mathbb{R}, x \text{ irrational}).$$

Then $\chi(a)$ is defined for any $a \in \mathbb{R}$ but $\lim_{x\to a} \chi(x)$ does not exist for any a. To see this, assume the contrary that $\lim_{x\to a} \chi(x) = L$ for some $L \in \mathbb{R}$. Given $\varepsilon = \frac{1}{3}$ there would exist $\delta > 0$ such that $|\chi(x) - L| < \frac{1}{3}$ if $0 < |x-a| < \delta$. But in the interval $(a, a+\delta)$ say, there is both a rational number and an irrational. If $x \in (a, a+\delta)$ is rational we would have $|1-L| < \frac{1}{3}$, while if $x \in (a, a+\delta)$ is irrational we would have $|0-L| < \frac{1}{3}$. A contradiction follows easily.

On the other hand , most of the functions that are "easy to write down" turn out to be continuous at all points where they are defined. For example, we proved that $\lim_{x\to 3}(x^2+2x)=15$. This shows that function f is defined by

$$f(x)=x^2+2x \quad (x \in \mathbb{R})$$

is continuous at $x=3$. For $f(3)=15$ and $\lim_{x\to 3} f(x)=15$. The next example in unit 2 shows that the function g defined by

$$g(x)=\sqrt{x+3} \qquad (0<x<2)$$

is continuous at $x=1$.

Theorem 1. If the real-valued functions f and g are continuous at $a \in \mathbb{R}$, then so are $f+g, f-g,$ and fg. If $g(a)\neq 0$, then f/g is also continuous at a.

Proof. Since f and g are continuous at a we have

$$\lim_{x\to a} f(x)=f(a) \text{ and } \lim_{x\to a} g(x)=g(a).$$

Then, by 4.1C, $\lim_{x\to a}[f(x)+g(x)]=f(a)+g(a)$. In other "words,"

$$\lim_{x\to a}(f+g)(x)=(f+g)(x)$$

This proves that $f+g$ is continuous at a. The remainder of the theorem is proved similarly.

A continuous function of a continuous function is continuous. More precisely,

Theorem 2. If f and g are real-valued functions, if f is continuous at a, and if g is continuous at $f(a)$, then $g\circ f$ is continuous at a.

Proof. We must show $\lim_{x\to a} g\circ f(x)=g\circ f(a)$ or,

$$\lim_{x\to a} g[f(x)]=g[f(a)].$$

That is, given $\varepsilon>0$ we must find $\delta>0$ such that

$$|g[f(x)]-g[f(a)]|<\varepsilon \quad (0<|x-a|<\delta). \qquad (1)$$

Let $b=f(a)$. Now by hypothesis

$$\lim_{x\to a} g[y]=g[b].$$

Hence there exists $\eta>0$ such that

$$|g[y] - g[b]| < \varepsilon \quad (|y - b| < \eta). \qquad (2)$$

But, also by hypothesis,

$\lim_{x \to a} f(x) = f(a).$

Thus (using η where we usually use ε) there exists δ such that

$$|f(x) - f(a)| < \eta \qquad (|x - a| < \delta),$$

$$|f(x) - b| < \eta \qquad (|x - a| < \delta). \qquad (3)$$

Thus if $|x - a| < \delta$ then $f(x)$ is within η of b and so we may substitute $f(x)$ for y in (2). Hence

$$|g[y] - g[b]| < \varepsilon \qquad (|y - b| < \eta).$$

Which implies (1), and the proof is complete.

3.3 REFORMULATION

We have defined "f is continuous at a" to mean $\lim_{x \to a} f(x) = f(a)$. That is, f is continuous at a if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon \quad (0 < |x - a| < \delta)$. However (as you were asked to observe in the last proof), the inequality $|f(x) - f(a)| < \varepsilon$ obviously holds if $x = a$. Thus, we need only write $|x - a| < \delta$ instead of $0 < |x - a| < \delta$. Here then, is a reformulation of definition.

Theorem 3. The real-valued function f is continuous at $a \in \mathbb{R}$, if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad (|x - a| < \delta)$$

Then, f is continuous at a if for any $\varepsilon > 0$ there exists $\delta > 0$ such that, if the distance from x to a is less than δ, then distance from $f(x)$ to $f(a)$ is less than ε. Show that the definition of continuity is based on the metric in $\mathbb{R}$.

Definition. If $a \in \mathbb{R}$, and $r > 0$ we define $B[a; r]$ to be the set of all $x \in \mathbb{R}$ whose distance to a is less than r. That is,

$$B[a; r] = \{x \in \mathbb{R} \,|\, |x - a| < r\}.$$

We call $B[a; r]$ the **open ball** of radius r about a.

It is clear that $B[a; r]$ is just a fancy way of denoting the bounded open interval $(a + r, a - r)$. However, in an arbitrary metric space there is no such thing as an interval. But the object $B[a; r]$ does have a counterpart in any metric space, which is the reason we defined it in terms of distance.

Thus reads "f is continuous at a if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x) \in B[f(a); \varepsilon]$ if $x \in B[a; \delta]$." That is, the entire open ball $B[a; \delta]$ is mapped by f into the open ball $B[f(a); \varepsilon]$.

Thus, f is continuous at a if and only if, for any open ball B about $f(a)$, there is an open ball about a which f maps entirely into B. It turns out to be more useful to be more useful to state this definition in terms of inverse images.

Theorem 4. The real-valued function f is continuous at $a \in \mathbb{R}$ if and only if the inverse image under f of any open ball $B[f(a); \varepsilon]$ about $f(a)$ contain an open ball $B[a; \delta]$ about a. (That is, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(B[f(a); \varepsilon]) \supset B[a; \delta]).$$

Our final reformulation of the continuity concept will be in terms of sequences, observe first that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a if and only if given $\varepsilon > 0$ there exists $N \in I$ such that $x_n \in B[a; \varepsilon]$ $(n \geq N)$.

That is, given any open ball B about a, all but a finite number of the x_n are in B.

Theorem 5. The real-valued function f is continuous at $a \in \mathbb{R}$ if and only if, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. That is, f is continuous at a if and only if

$$\lim_{n\to\infty} x_n = a \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) = f(a) \qquad (*).$$

Proof. Let us first assume that f continuous at a and prove that (*) holds. Let $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a. [Then $f(x_n)$ will be defined for n sufficiently large.] We must show that $\lim_{n\to\infty} f(x_n) = f(a)$ that is, given $\varepsilon > 0$ there exists $N \in I$ such that

$$f(x_n) \in B[f(a); \varepsilon] \qquad (n \geq N). \qquad (1)$$

But since f is continuous at $a \in \mathbb{R}$ there exists $\delta > 0$ such that

$$f(x) \in B[f(a); \varepsilon] \qquad (x \in B[a; \delta]). \qquad (2)$$

Furthermore, since $\lim_{n\to\infty} x_n = a$, there exists $N \in I$ such that

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$$x_n \in B[a;\delta] \qquad (n \geq N). \qquad (3)$$

.

For this N, (1) follows from (2) and (3).

Conversely, suppose (*) holds. We must prove that f is continuous at a. Assume the contrary. Then, for some $\varepsilon > 0$ the inverse image under f of $B = B[f(a);\varepsilon]$ contains no open ball about a. In particular, $f^{-1}(B)$ does not contain $B[a;\frac{1}{n}]$ for any $n \in I$. Thus, for each $n \in I$, there is point $x_n \in B[a;\frac{1}{n}]$ such that $f(x_n) \notin B$. That is

$$|x_n - a| < \frac{1}{n} \text{ but } |f(x_n) - f(a)| \geq \varepsilon.$$

This clearly contradicts (*), so f must be continuous at a.

$$\lim_{n\to\infty} g[f(x_n)] = g[f(a)]$$

Where $\{x_n\}_{n=1}^{\infty}$ is any sequence of real numbers such that

$$\lim_{n\to\infty} x_n = a.$$

Since f is continuous at a, imply (1) and the proof is contradict.

3.4 BOUNDED SETS IN METRIC SPACE

Definition. Let (M,d) be a metric space. We say that a subset A of M is **bounded** if there exists a positive real number k such that $d(x,y) \leq k$ for all $x,y \in A$.

Definition. Let (M,d) be a metric space. Let $A \subseteq M$. Then **the diameter of A**, denoted by $d(A)$, is defined by $d(A) = l.u.b\{x \in A | x,y \in A\}$.

Note 1. A non-empty set A is a bounded set iff $d(A)$ is finite.

Note 2. Let $A,B \subseteq M$. Then $A \subseteq B \Rightarrow d(A) \leq d(B)$.

3.5 PROBLEMS

Example 1. Any finite subset A of a metric space (M,d) is bounded.

Proof. Let A be any finite subset of M.

If $A = \Phi$ then A is obviously bounded.

Let $A \neq \Phi$. Then $\{d(x,y)|x,y \in A\}$ is a finite set of real numbers. Let $k = \max\{d(x,y)|x,y \in A\}$. Clearly $d(x,y) \leq k$ for all $x,y \in A$.

$\therefore A$ is bounded.

Example 2. $[0,1]$ is a bounded subset of $\mathbb{R}$ with usual metric since $d(x,y) \leq 1$ for all $x,y \in [0,1]$.

More generally any finite interval and any subset of $\mathbb{R}$ which is contained in a finite interval are bounded subsets of $\mathbb{R}$.

Example 3. $(0,\infty)$ is a unbounded subset of $\mathbb{R}$.

Example 4. If consider $\mathbb{R}$ with discrete metric, then $(0,\infty)$ is a bounded subset of $\mathbb{R}$, since $d(x,y) \leq 1$ for all $x,y \in (0,\infty)$.

More generally any subset of a discrete metric space M is bounded subsets of M.

Example 5. In l_2 let $e_1 = \{1,0,\dots,0,\dots\}, e_2 = \{0,1,0,\dots,0,\dots\}, e_3 = \{0,0,1,\dots,0,\dots\}, \dots.$.

Let $A = \{e_1, e_2, \dots, e_n, \dots\}$.

Then A is a bounded subset of l_2.

Proof. $d(e_n, e_m) = \begin{cases} \sqrt{2} & if\ n \neq m \\ 0 & if\ n = m. \end{cases}$

$\therefore d(e_n, e_m) \leq \sqrt{2}$ for all $e_n, e_m \in A$.

$\therefore A$ is a bounded set in l_2.

Example 6. Let (M,d) be a metric space. Define $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$.

We know that (M,d_1) is also a metric space.

Also $d_1(x,y) < 1$ for all $x,y \in M$.

Hence (M,d_1) is a bounded metric space.

NOTES

Example 7. The diameter of any non-empty subset in a discrete metric space is 1.

CHECK YOUR PROGRESS

1. $d(\Phi)$?
2. Define diameter
3. Describe length of an interval.

3.6 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. In any metric space, $d(\Phi) = -\infty$.

2. Let (M, d) be a metric space. Let $A \subseteq M$. Then the **diameter** of A, denoted by $d(A)$, is defined by $d(A) = l.u.b\{x \in A | x, y \in A\}$.

3. In $\mathbb{R}$ the diameter of any interval is equal to the length of the interval. For example the diameter of [0,1] is 1.

3.7 SUMMARY

1. The function f is **continuous at** $\boldsymbol{a} \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$.
2. If the real-valued functions f and g are continuous at $a \in \mathbb{R}$, then so are $f + g, f - g,$ and fg. If $g(a) \neq 0$, then f/g is also continuous at a.
3. If f and g are real-valued functions, if f is continuous at a, and if g is continuous at $f(a)$, then $g \circ f$ is continuous at a.
4. If $a \in \mathbb{R}$, and $r > 0$ we define $B[a; r]$ to be the set of all $x \in \mathbb{R}$ whose distance to a is less than r. That is, $B[a; r] = \{x \in \mathbb{R} | |x - a| < r\}$. We call $B[a; r]$ the **open ball** of radius r about a.

5. Let (M, d) be a metric space. Let $A \subseteq M$. Then **the diameter of A**, denoted by $d(A)$, is defined by $d(A) = l.u.b\{x \in A | x, y \in A\}$.
6. A non-empty set A is a bounded set iff $d(A)$ is finite.

3.8 KEYWORDS

1. **Continuous:** The function f is continuous at $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$.
2. **Open ball:** If $a \in \mathbb{R}$, and $r > 0$ we define $B[a; r]$ to be the set of all $x \in \mathbb{R}$ whose distance to a is less than r. That is, $B[a; r] = \{x \in \mathbb{R} | |x - a| < r\}$. We call $B[a; r]$ the open ball of radius r about a.
3. **Bounded:** Let (M, d) be a metric space. We say that a subset A of M is bounded if there exists a positive real number k such that $d(x, y) \leq k$ for all $x, y \in A$.
4. **Diameter:** Let (M, d) be a metric space. Let $A \subseteq M$. Then the diameter of A, denoted by $d(A)$, is defined by $d(A) = l.u.b\{x \in A | x, y \in A\}$.
5. **Usual metric:** A metric space denoted by $(\mathbb{R}, d)$ is defined by $d(x, y) = |x - y|$. This metric d is called the usual metric for $\mathbb{R}$.
6. **Discrete metric:** Any non-empty set M we define d as $d(x, y) = \begin{cases} 0 \ if \ x = y \\ 1 \ if \ x \neq y. \end{cases}$ Then d is a metric on M. This is called the discrete metric on M.
7. **n-dimensional Euclidean space:** $\mathbb{R}^n$ with usual metric is called the n-dimensional Euclidean space.

3.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Let (M, d) is a metric space. Define $d_1(x, y) = \min\{d(x, y), 1\}$. Prove that (M, d_1) is a bounded metric space.
2. Prove that in a metric space any subset of a bounded set is bounded.
3. In $\mathbb{R}$, with usual metric find $B(1,1)$.
4. In $\mathbb{R}^2$, with usual metric find $B((0,0), \frac{1}{2})$.

8. Determine $\left(-\frac{1}{2},\frac{1}{2}\right) \cup \{1\}$ is open in $\mathbb{R}$ with usual metric.

NOTES

3.10 FURTHER READINGS

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2. Richard R. Goldberg, Methods of Real Analysis, Oxford & IBH Publishing Company, New Delhi.
3. D. Somasundaram & B. Choudhary, A first course in Mathematical Analysis, Narosa Publishing House, Chennai.
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UNIT-4 SUBSPACES

NOTES

Structure

4.0 INTRODUCTION

In mathematics, a metric space aimed at is subspace is a categorical construction that has a direct geometric meaning. It is also a useful step toward the construction of the metric envelops, or tight span, which are basic objects of the category of metric spaces.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by subspaces.
- Determine if subsets of a metric space are open, closed sets.
- Discuss limit point, closure and dense set.

4.2 SUBSPACE

Definition. Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with the same metric d. we say that (M_1, d) is a **subspace** of (M, d).

NOTES

Note. If M_1 is a subspace of M a set which is open in M_1 need not be open in M.

For example, if $M = \mathbb{R}$ with usual metric and $M_1 = [0,1]$ then $[0, \frac{1}{2})$ is open in M_1 but not open in M.

We now proceed to investigate the nature of open sets in subspace M_1 of a metric space M.

Theorem 1. Let M be a metric space and M_1 a subspace of M. Let $A_1 \subseteq M_1$. Then A_1 is open in M_1 iff there exists an open set A in M such that $A_1 = A \cap M_1$.

Proof. Let M_1 be a subspace of M. Let $a \in M_1$.

We denote $B_1(a,r)$ the open ball in M_1 with center a, radius r.

Then $B_1(a,r) = \{x \in M_1 | d(a,x) < r\}$.

Also, $B(a,r) = \{x \in M | d(a,x) < r\}$.

Hence, $B_1(a,r) = B(a,r) \cap M_1$. ....(1)

Now, let A_1 be an open set in M_1.

$A_1 = \bigcup_{x \in A_1} B(x, r(x))$

$= \bigcup_{x \in A_1} [B(x, r(x)) \cap M_1]$ (by (1))

$= \left[\bigcup_{x \in A_1} B(x, r(x))\right] \cap M_1$.

$= A \cap M_1$ where $A = \bigcup_{x \in A_1} B(x, r(x))$ which is open M.

Conversely, let $A_1 = A \cap M_1$ where A is open in M.

We claim that A_1 is open in M_1.

Let $x \in A_1$.

$\therefore x \in A$ and $x \in M_1$.

Since A is open in M there exists a positive real number r such that $B(x,r) \subseteq A$.

$\therefore M_1 \cap B(x,r) \subseteq M_1 \cap A$.

NOTES

i.e. $B_1(x,r) \subseteq A_1$ (using (1))

$\therefore A_1$ is open in M_1 .

Example 1. Let $M = \mathbb{R}$ and $M_1 = [0,1]$. Let $A_1 = [0,\frac{1}{2})$.

Now $A_1 = \left[0,\frac{1}{2}\right) = \left(\frac{1}{2},-\frac{1}{2}\right) \cap [0,1]$ and $\left(\frac{1}{2},-\frac{1}{2}\right)$ is open in $\mathbb{R}$.

$\therefore [0,\frac{1}{2})$ is open in $[0,1]$.

SOLVED PROBLEMS

Problem 1. Let M_1 be a subspace of a metric space M. Prove that every open set A_1 of $M_1$is open in M iff is open in M.

Solution. Suppose that every open set A_1 of M_1 is open in M.

Now, M_1 is open in M_1.

Hence M_1 is open in M.

Conversely, suppose M_1 is open in M.

Let A_1 be an open set in M_1.

Then by theorem 1, there exists an open set A in M such that $A_1 = A \cap M_1$.

Since A and M_1 are open in M_1we get A_1 is open in M.

4.3 INTERIOR OF A SET

Definition. Let (M,d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an **interior** of A if there exists a positive real number r such that $B(x,r) \subseteq A$.

The set of all interior points of A is called the interior of A and it is denoted by $Int\ A$.

Note. $Int\ A \subseteq A$.

Example 1. Consider $\mathbb{R}$ with usual metric.

NOTES

(a) Let $A = [0,1]$. Clearly 0 and 1 are not interior points of A and any point $x \in (0,1)$ is an interior point of A. Hence $Int\ A = (0,1)$.

(b) Let $A = \mathbb{Q}$. Let $x \in \mathbb{Q}$Then for any positive real number r, $B(x,r) = (x-r, x+r)$ contains irrational numbers.

$\therefore B(x,r)$ is not a subset of $\mathbb{Q}$.

$\therefore x$ is not an interior point of $\mathbb{Q}$

Since $x \in \mathbb{Q}$ is arbitrary, no point of $\mathbb{Q}$ is an interior point of $\mathbb{Q}$.

$$\therefore Int\ \mathbb{Q} = \Phi$$

(c) Let A be a finite subset $\mathbb{R}$. Then $Int\ A = \Phi$.

(d) Let $A = \left\{0,1,\frac{1}{2},\dots\dots,\frac{1}{n},\dots\dots\right\}$. Then $Int\ A = \Phi$.

Example 2. Consider $\mathbb{R}$ with discrete metric.

Let $A = [0,1]$. Let $x \in [0,1]$.

Then $B\left(x,\frac{1}{2}\right) = \{x\} \subseteq A$

$\therefore x$ is an interior point of A.

Since $x \in [0,1]$ is arbitrary $Int\ A = A$.

Basic properties of interior are given in the following theorem.

Theorem 2. Let (M,d) be a metric space. Let $A, B \subseteq M$.

(i) A is open iff A=Int A. In particular $Int\ \Phi = \Phi$ and $Int\ \mathrm{M} = \mathrm{M}$.

(ii) Int A = Union of all open sets contained in A.

(iii) $Int\ A$ is an open subset of A and if B is any other open set contained in A then $B \subseteq Int\ A$ .i.e. Int A is the largest open set contained in A.

(iv) $A \subseteq B \Rightarrow Int\ A \subseteq Int\ B$.

(v) $Int\ (\mathrm{A} \cap \mathrm{B}) = \mathrm{Int\ A} \cap \mathrm{Int\ B}$.

(vi) $Int\ (\mathrm{A} \cup \mathrm{B}) \supseteq \mathrm{Int\ A} \cup \mathrm{Int\ B}$.

Proof.

(i) Follows from the definitions of open set.

(ii) Let $G = \cup\{B | B\ is\ an\ open\ subset\ of\ A\}$.

NOTES

To prove that $Int\ A = G$.
Let $x \in Int\ A$.
$\therefore$ There exists a positive real number r such that
$B(x,r) \subseteq A$.
Thus $B(x,r)$ is an open set contained in A.
$\therefore B(x,r) \subseteq G$.
$\therefore x \in G$.

$\therefore Int\ A \subseteq G$. ......(1)

Now, let $x \in G$.

Then there exists an open set B such that $x \in B$ and $B \subseteq A$.

Now, since B is open and $x \in B$ there exists a positive real number r such that $B(x,r) \subseteq B \subseteq A$.

$\therefore x$ is an interior point of A.

Hence $G \subseteq Int\ A$. ....(2)

From (1) and (2), we get $G = Int\ A$.

(iii) Since union of any collection of open sets is open

(ii)$\Rightarrow$ $Int\ A$ is an open set.

Trivially $Int\ A \subseteq A$.

Now, let B be any open set contained in A.

Then B⊆G=Int A. (by 2)

$\therefore$ $Int\ A$ is the largest open set contained in A.

(iv) Let $x \in Int\ A$.
$\therefore$ There exists a real number r>0 such that B(x,r)⊆A.
But $A \subseteq B$. Hence $B(x,r) \subseteq B$.
$\therefore x \in Int\ B$. Hence Int A ⊆ Int B.

(v) $A \cap B \subseteq A$.
$\therefore Int\ (A \cap B) \subseteq Int\ A$. (by (iv))
Similarly $Int\ (A \cap B) \subseteq Int\ B$.
$\therefore Int\ (A \cap B) \subseteq Int\ A \cap Int\ B$. ....(1)
Now, $Int\ A \subseteq A$; $IntB \subseteq B$.
Hence $Int\ A \cap Int\ B \subseteq A \cap B$.
Thus $Int\ A \cap Int\ B$ is an open set contained in $A \cap B$.
But $Int\ (A \cap B)$ is the largest open set contained in $A \cap B$.

$\therefore Int\ A \cap Int\ B \subseteq Int\ (A \cap B)$

NOTES

From (1) and (2) we get $Int\ (A \cap B) = Int\ A \cap Int\ B$.

(vi) $A \subseteq A \cup B$.

$\therefore Int\ A \subseteq\ (A \cup B)$ (by (iv))

Similarly, $Int\ A \subseteq\ (A \cup B)$

$$\therefore Int\ A \cup Int\ B \subseteq\ Int\ (A \cup B).$$

Note. $Int\ (A \cup B)$ need not be equal to $Int\ A \cup Int\ B$.

For example, in $\mathbb{R}$ with usual metric consider $A = (0,2]$ and $B = (2,3)$.

But, $Int\ A \cup Int\ B = (0,2) \cup (2,3) = (0,3) - \{2\}$.

$\therefore Int\ (A \cup B) \neq IntA \cup IntB$.

4.4 OPEN SET

Definitions. Let (M, d) be a metric space. Let A be a subet of M. Then A is said to be **open** in M if for every $x \in A$ there exists a positive real number r such that $B(x, r) \subseteq A$.

Example 1. In $\mathbb{R}$ with usual metric (0,1) is an open set.

Proof. Let $x \in (0,1)$.

Choose $r = \min\{x - 0, 1 - x\} = \min\{x, 1 - x\}$.

Clearly $r > 0$ and $B(x, r) =\ (x - r, x + r) \subseteq (0,1)$.

$\therefore$ (0,1) is open.

Example 2. In $\mathbb{R}$with usual metric [0,1) is not open since no open ball with center 0 is contained [0,1).

Example 3. Consider $M = [0,2)$ with usual metric. Let $A = [0,1) \subseteq M$. Then A is open in M.

Proof. Let $x \in [0,1)$.

If $x = 0$ then $B\left(0, \frac{1}{2}\right) = \left[0, \frac{1}{2}\right) \subseteq A$.

If $x \neq 0$ choose $r = \min\{x, 1 - x\}$.

Clearly $r > 0$ and $B(x, r) =\ (x - r, x + r) \subseteq (0,1)$.

$\therefore A$ is open in M.

Example 4. Any open interval (a, b) is an open set in $\mathbb{R}$ with usual metric.

Proof. Let $x \in (a, b)$.

Let $r = \min\{x - a, b - x\}$.

Then $B(x, r) \subseteq (a, b)$. Hence (a, b) is an open set.

NOTES

Note. Similarly we can prove that $(-\infty, a)$ and (a, ∞) are open sets.

Example 5. In $\mathbb{R}$ with usual metric any finite non-empty subset A of $\mathbb{R}$ is not an open set.

Proof. Any open ball in $\mathbb{R}$ is a bounded open interval which is an infinite subset of $\mathbb{R}$. Hence it cannot be contained in the finite subset A. Hence A is not open in $\mathbb{R}$.

Example 6. $\mathbb{Q}$ is not open in $\mathbb{R}$.

Proof. Let $x \in \mathbb{Q}$. Then for any $r > 0$ the interval $(x - r, x + r)$ contains both rational and irrational numbers.

$\therefore$ $(x - r, x + r)$ is not a subset of $\mathbb{Q}$.

$\therefore$ $\mathbb{Q}$ is not open in $\mathbb{R}$.

Example 7. $\mathbb{Z}$ is not open in $\mathbb{R}$.

Proof. Let $x \in \mathbb{Z}$. Then for any $r > 0$ the interval $(x - r, x + r)$ is not a subset of $\mathbb{Z}$. Hence $\mathbb{Z}$ is not open in $\mathbb{R}$.

Theorem 3. In any metric space M.

(i) Φ is open.
(ii) M is open.

Proof. (i) Trivially Φ is an open set.

(ii)Let $x \in M$. Clearly for any $r > 0$ $B(x, r) \subseteq M$. Hence M is an open set.

Theorem 4. In any metric space (M, d) each open ball is an open set.

Proof. Let $B(a, r)$ be an open ball in M.

Let $x \in B(a, r)$.

Then $d(a, x) < r$.

NOTES

$\therefore r - d(a,x) > 0.$

Let $r_1 = r - d(a,x)$.

We claim that $B(x, r_1) \subseteq B(a, r)$.

Let $y \in B(x, r_1)$

$\therefore d(x,y) < r_1 = r - d(a,x).$

$\therefore d(x,y) + d(a,x) < r$ (1)

Now, $d(a,y) < d(a,x) + d(x,y) < r$ (by (1)).

$\therefore d(a,y) < r.$

$\therefore y \in B(a,r)$

Hence $B(x, r_1) \subseteq B(a,r)$.

$\therefore B(a,r)$ is an open set.

Theorem 5. In any metric space the union of any family of open sets is open.

Proof. Let (M, d) be a metric space.

Let $\{A_i | i \in I\}$ be a family of open sets in M.

Let $A = \bigcup_{i \in I} A_i$

If $A = \Phi$ then A is open.

Therefore, let $A \neq \Phi$. Let $x \in A$.

Then $x \in A_i$ for some $i \in I$.

Since A_i is open there exist an open ball $B(x,r)$ such that $B(x,r) \subseteq A_i$.

$\therefore B(x,r) \subseteq A.$

Hence A is open.

Theorem 6. In any metric space the intersection of a finite number of open sets is open.

Proof. Let (M, d) be a metric space.

Let $A_1, A_2, \dots, A_n, \dots$. be open sets in M.

Let $A = A_1 \cap A_2 \cap \dots \cap A_n \cap \dots$.

NOTES

If $A = \Phi$. Let $x \in A$.

$\therefore x \in A_i$ for each $i = 1,2, \dots, n$.

Since each A_i is an open set there is a positive real number r_i such that
$B(x, r_i) \subseteq A_{i.}$ (1)

Let $r = \min\{r_1, r_2, \dots, r_n\}$.

Obviously r is a positive real number and $B(x, r) \subseteq B(a, r_i)$ for all $i = 1,2, \dots, n$.

Hence $B(x, r) \subseteq A_i$ for all $i = 1,2, \dots, n$. (by 1)

$\therefore B(x, r) \subseteq \bigcap_{i=1}^{n} A_i$.

$\therefore B(x, r) \subseteq A$.

$\therefore A$ is open.

Note. The intersection of an infinite number of open sets in a metric space need not be open.

For example, consider $\mathbb{R}$ with usual metric.

Let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.

Then A_n is open in $\mathbb{R}$ for all n. (refer example 4)

But $\bigcap_{i=1}^{\infty} A_n = \{0\}$ which is not open in $\mathbb{R}$. (refer example 5)

We now give a characterization of open sets in terms of open balls.

Theorem 7. Let (M, d) be a metric space. Let A be any non-empty subset of M. Then A is open iff A can be expressed as the union of a family of open balls.

Proof. Let A be a non-empty open subset of M.

Let $x \in A$.

Since A is an open set there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \subseteq A$.

Clearly $\bigcup_{x \in A} B(x, r_x) = A$.

Thus A is the union of a family of open balls.

NOTES

Conversely, let A be a union of open balls.

Then A is open.

SOLVED PROBLEMS

Example 1. Let (M, d) be a metric space. Let x, y be two distinct points of M. Prove that there exist disjoint open balls with centers x and y respectively.

Solution. Since $x \neq y$, $d(x, y) = r > 0$.

Consider the open balls $B(x, \frac{1}{4}r)$ and $B\left(y, \frac{1}{4}r\right)$.

We claim that $B(x, \frac{1}{4}r) \cap B(y, \frac{1}{4}r) = \Phi$

Suppose $B(x, \frac{1}{4}r) \cap B(y, \frac{1}{4}r) \neq \Phi$

Let $z \in B\left(x, \frac{1}{4}r\right) \cap B\left(y, \frac{1}{4}r\right)$.

$\therefore z \in B(x, \frac{1}{4}r)$ and $z \in B(y, \frac{1}{4}r)$.

$\therefore d(x, z) < \frac{1}{4}r$ and $d(y, z) < \frac{1}{4}r$.

Now, $d(x, y) \leq d(x, z) + d(z, y)$.

$\therefore r \leq \frac{1}{4}r + \frac{1}{4}r = \frac{1}{2}r$

Which is a contradiction.

Hence $B\left(x, \frac{1}{4}r\right) \cap B\left(y, \frac{1}{4}r\right) = \Phi$.

Example 2. Let (M, d) be a metric space. Let $x \in M$. Show that $\{x\}^c$ is open.

Solution. Let $y \in \{x\}^c$. Then $y \neq x$.

$\therefore d(x, y) = r > 0$.

Cleary $B(y, \frac{1}{2}r) \subseteq \{x\}^c$.

$\therefore \{x\}^c$ is open.

Example 3. Let (M, d) be a metric space. Show that every subset of M is open iff $\{x\}$ is open for all $x \in M$.

Solution. Suppose every subset of M is open.

Then obviously $\{x\}$ be open for all $x \in M$.

NOTES

Conversely, let $\{x\}$ be open for all $x \in M$.

Let A be any subset of M.

If $A = \Phi$ then A is open.

Let $A \neq \Phi$. Then $A = \bigcup_{x \in A}\{x\}$.

By hypothesis $\{x\}$ is open.

Hence A, is open.

Example 4. Let $A = \left\{(a_n) \middle| (a_n) \in l_2 \ and \ [\sum_{n=1}^{\infty} a_n^2]^{1/2} < 1\right\}$. Prove that A is an open subset of l_2.

Solution. We first prove that $A = B(0,1)$ where $0 = (0,0,0, \ldots .)$

Let $x \in A$. Hence $[\sum_{n=1}^{\infty} x_n^2]^2 < 1$.

$\therefore d(x, 0) = [\sum_{n=1}^{\infty}(x_n - 0)^2]^{1/2} = [\sum_{n=1}^{\infty} x_n^2]^{1/2} < 1$

Thus $d(x, 0) < 1$

$\therefore x \in B(0,1)$

$\therefore A \subseteq B(0,1)$ (1)

Now, let $y \in B(0,1)$

$\therefore d(0, y) < 1$.

$\therefore [\sum_{n=1}^{\infty}(y_n - 0)^2]^{1/2} < 1$

$\therefore [\sum_{n=1}^{\infty}(y_n)^2]^{1/2} < 1$

$\therefore y \in A$.

$\therefore B(0,1) \subseteq A$. (2)

By (1) and (2) we get $A = B(0,1)$

Now, the open ball $B(0,1)$ is an open set.

$\therefore A$ is an open set.

Example 5. Prove that any open subset of R can be expressed as the union of a countable number of mutually disjoint open intervals.

Solution. Let A be an open subset of R. let $x \in A$. Then there exists a positive real number r such that $B(x, r) = (x - r, x + r) \subseteq A$.

NOTES

Thus there exist an open interval I such that $x \in I$ and $I \subseteq A$

Let I_x be denote the largest open interval such that $x \in I$ and $I_x \subseteq A$

Clearly $\bigcup_{x \in A} I_x = A$.

Now let $x, y \in A$.

We claim that $I_x = I_y$ or $I_x \cap I_y = \Phi$

Suppose $I_x \cap I_y \neq \Phi$

Then $I_x \cup I_y$ is an open interval contained in A.

But I_x is the largest open interval such that $x \in I_x$ and $I_x \subseteq A$.

$\therefore I_x \cup I_y = I_x$ so that $I_y \subseteq I_x$

Similarly $I_x \subseteq I_y$.

$\therefore I_x = I_y$. Thus the intervals I_x are mutually disjoint.

We claim that the set $F = \{I_x | x \in A\}$ is countable.

Now for each $I_x \in F$ choose a rational number $r_x \in I_x$.

Since the intervals I_x are mutually disjoint $I_x = I_y \Rightarrow r_x \neq r_y$.

$\therefore f: F \to \mathbb{Q}$ defined by $f(I_x) = r_x$ is 1-1.

$\therefore F$ is equivalent to a subset of $\mathbb{Q}$ which is countable.

$\therefore F$ is countable.

Definition. Let d and ρ be the two metrics on M. Then the metrics d and ρ are said to be **equivalent** if the open sets of (M, ρ) are the open sets of (M, d) and conversely.

Example 6. Let (M, d) be a metric space. Define $\rho(x, y) = 2d(x, y)$. Then d and ρ are equivalent metrics.
Solutions. We know that ρ is a metric on M.

We first prove that $B_d(a, r) = B_\rho(a, 2r)$

Let $x \in B_d(a, r)$

$\therefore d(a, x) < r$.

$\therefore 2d(a, x) < 2r$.

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$\therefore \rho(a,x) < 2r$. Hence $x \in B_\rho(a,2r)$

$\therefore B_d(a,r) \subseteq B_\rho(a,2r)$ ....(1)

Now, let $x \in B_\rho(a,2r)$

$\therefore \rho(a,x) < 2r$.

$\therefore \frac{1}{2}\rho(a,x) < r$.

$\therefore d(a,x) < r$. Hence $x \in B_d(a,r)$.

$B_\rho(a,2r) \subseteq B_d(a,r)$ ....(2)

$\therefore$By (1) and (2) we get $B_d(a,r) = B_\rho(a,2r)$. .....(3)

Now, let G be any open subset in (M,d). Let $\in G$. Hence there exists $r > 0$ such that $B_d(a,r) \subseteq G$.

$\therefore$ $B_\rho(a,2r) \subseteq G$.

$\therefore G$ is open in (M,ρ).

Conversely, suppose G is open in (M,ρ).

Let $a \in G$. Hence there exists r>0 such that $B_\rho(a,r) \subseteq G$.

Hence $B_d(a,\frac{1}{2}r) \subseteq G$ (using 3). Hence G is open in (M,d).

$\therefore d$ and ρ are equivalent metrics.

Example 7. Let (M,d) be a metric space. Define $(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d and ρ are equivalent metrics on M.

Solution. We know that ρ is a metric on M.We first prove $B_\rho(a,r) = B_d(a,\frac{r}{1-r})$ provided $0 < r < 1$.

Let $x \in B_\rho(a,r)$. Hence $\rho(a,x) < r$.

$\therefore \frac{d(a,x)}{1+d(a,x)} < r$.

$\therefore d(a,x) < r[1 + d(a,x)]$.

$\therefore d(a,x)[1-r] < r$.

$\therefore d(a,x) < \frac{r}{[1-r]}$ (since $0 < r < 1$)

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$\therefore x \in B_d\left(a, \frac{r}{1-r}\right)$.

$\therefore B_\rho(a,r) \subseteq B_d\left(a, \frac{r}{1-r}\right)$. ... . (1)

Now, let $x \in B_d\left(a, \frac{r}{1-r}\right)$. Hence $d(a,x) < \frac{r}{1-r}$.

$\therefore d(a,x)[1-r] < r$

$\therefore d(a,x) < r[1 + d(a,x)]$.

$\therefore \frac{d(a,x)}{1+d(a,x)} < r$.

$\therefore \rho(a,x) < r$.

$\therefore x \in B_\rho(a,r)$.

$\therefore B_d\left(a, \frac{r}{1-r}\right) \subseteq B_\rho(a,r)$
. ... . (2)

$\therefore$ By (1) and (2) we get $B_d\left(a, \frac{r}{1-r}\right) = B_\rho(a,r)$.
... . . (3)

Now, let G be open in (M,ρ).

Let $a \in G$. Hence there exists $r > 0$ such that $B_\rho(a,r) \subseteq G$.

Without loss of generality we may assume that $r < 1$.

$\therefore B_d\left(a, \frac{r}{1-r}\right) \subseteq G$ (By (3)).

$\therefore G$ is open in (M,d).

Conversely, let G be open in (M,d).

$\therefore$ There exists $r > 0$ such that $B_d(a,r) \subseteq G$.

$\therefore B_d\left(a, \frac{r}{1-r}\right) \subseteq G$ (using 3).

$\therefore G$ is open in (M,ρ).

Hence d and ρ are equivalent metrics.

Example 8. If d and ρ are metrics on M and if there exists $k > 1$ such that $\frac{1}{k}\rho(x,y) \leq d(x,y) \leq k\rho(x,y)$ for all $x, y \in M$. Prove that d and ρ are equivalent metrics.

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Solution. Suppose there exists $k > 1$ such that for all $x, y \in M$

$\frac{1}{k}\rho(x,y) \leq d(x,y) \leq k\rho(x,y)$ ....(1)

Let G be an open set in (M,d).

Let $a \in G$. Hence there exists $r > 0$ such that $B_d(a,r) \subseteq G$.

We now claim that $B_\rho\left(a,\frac{r}{k}\right) \subseteq G$. ....(2)

Let $x \in B_\rho\left(a,\frac{r}{k}\right)$.

$\therefore\ \rho(a,x) < \frac{r}{k}$.

$\therefore\ k\rho(a,x) < r$.

$\therefore\ \rho(a,x) < r$. (using 1)

$\therefore x \in B_d(a,r) \subseteq G$ (by 2)

$\therefore x \in G$. Hence $B_\rho(a,\frac{r}{k}) \subseteq G$.

$\therefore G$ is open in (M,ρ)

Conversely, let G be open in (M,ρ). Let $a \in G$.

$\therefore$ There exists $r > 0$ such that $B_\rho(a,r) \subseteq G$.
....(3)

We claim that $B_d\left(a,\frac{r}{k}\right) \subseteq G$.

$\therefore x \in B_d\left(a,\frac{r}{k}\right)$.

$\therefore\ d(a,x) < \frac{r}{k}$.

$\therefore\ kd(a,x) < r$.

$\therefore\ \rho(a,x) < r$. (using 1)

$\therefore x \in B_\rho(a,r) \subseteq G$ (by 3)

$\therefore x \in G$. Hence $B_d(a,\frac{r}{k}) \subseteq G$.

Hence G is open in (M,d).

$\therefore d$ and ρ are equivalent metrics.

NOTES

4.5 CLOSED SETS

Definition. Let (M, d) be a metric space. Let $A \subseteq M$. Then A is said to be **closed** in M if the complement of A is open in M.

Example 1. In $\mathbb{R}$ with usual metric any closed interval $[a, b]$ is closed set.

Proof. $[a, b)$ is not open in $\mathbb{R}$ since a is not an interior point of $[a, b)$.

Now, $[a, b)^c = \mathbb{R} - [a, b) = (-\infty, a) \cup (b, \infty)$.

Also $(-\infty, a)$ and (b, ∞) are open in $\mathbb{R}$.

i.e. $[a, b]^c$ is open in $\mathbb{R}$.

$\therefore$ $[a, b]$ is closed in $\mathbb{R}$.

Example 2. In $\mathbb{R}$ with usual metric $[a, b)$ is neither closed nor open.

Proof. $[a, b)$ is not open in $\mathbb{R}$ since a is not an interior point of $[a, b)$.

Now, $[a, b)^c = \mathbb{R} - [a, b) = (-\infty, a) \cup [b, \infty)$ and this set is not open since b is not an interior point.

$\therefore$ $[a, b)$ is not closed in $\mathbb{R}$.

Hence $[a, b)$ is neither open not closed in $\mathbb{R}$.

Example 3. In $\mathbb{R}$ with usual metric $(a, b]$ is neither closed nor open.

Proof is similar to example 2.

Example 4. $\mathbb{Z}$ is closed.

Proof. $\mathbb{Z}^c = \bigcup_{n=-\infty}^{\infty}(n, n+1)$.

The open interval $(n, n+1)$ is open and union of open sets is open.

$\mathbb{Z}^c$ is open. Hence $\mathbb{Z}$ is closed.

Example 5. $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Proof. $\mathbb{Q}^c$ = the set of irrationals which is not open in $\mathbb{R}$.

Therefore, $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Example 6. The set of irrational numbers is not closed in $\mathbb{R}$.

Proof is similar to that of example 5.

Example 7. In $\mathbb{R}$ with usual metric every singleton set is closed.

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Proof. Let $a \in \mathbb{R}$.

Then $\{a\}^c = \mathbb{R} - \{a\} = (-\infty, a) \cup (a, \infty)$.

Since $(-\infty, a)$ and (a, ∞) are both open sets $(-\infty, a) \cup (a, \infty)$ is open.

$\therefore \{a\}^c$ is open $\mathbb{R}$. Hence $\{a\}$ is closed in $\mathbb{R}$.

Definition. Let (M, d) be a metric space. Let $a \in M$. Let r be any positive real number. Then the **closed ball or the closed sphere** with center a and radius r, denoted by $B_d[a, r]$, is defined by

$$B_d[a, r] = \{x \in M | d(a, x) \leq r\}.$$

When the metric d under consideration is clear we write $B[a, r]$ instead of $B_d[a, r]$.

Example 1. In $\mathbb{R}$ with usual metric $B[a, r] = [a - r, a + r]$.

Example 2. In $\mathbb{R}^2$ with usual metric let $a = (a_1, a_2) \in \mathbb{R}^2$.

Then $B[a, r] = \{(x, y) \in \mathbb{R}^2 | (a_1, a_2), (x, y) \leq r\}$.

$$= \{(x, y) \in \mathbb{R}^2 | (x - a_1)^2 + (y - a_2)^2 \leq r^2\}.$$

Hence $B[a, r]$ is the set of all points which lie within and on the circumference of the circle with center a and radius r.

Theorem 8. In any metric space every closed ball is a closed set.

Proof. Let (M, d) be a metric space.

Let $B[a, r]$ be a closed ball in M.

Case (i). Suppose $B[a, r]^c = \Phi$.

$\therefore B[a, r]^c$ is open and hence $B[a, r]$ is closed.

Case (ii). Suppose $B[a, r]^c \neq \Phi$.

Let $x \in B[a, r]^c$.

$\therefore x \notin B[a, r]^c$.

$\therefore d(a, x) > r$.

$\therefore d(a, x) - r > 0$.

Let $r_1 = d(a, x) - r$.

We claim that $B(x, r_1) \subseteq B[a, r]^c$.

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Let $y \in B(x, r_1)$.

Then $d(x, y) < r_1 = d(a, x) - r$.

$$\therefore d(a, x) > d(x, y) - r.$$

Now, $d(a, x) \leq d(a, y) + d(y, x)$.

$$\therefore d(a, y) \geq d(a, x) - d(y, x).$$

$> d(x, y) + r - d(y, x)$ (by 1).

$$= r.$$

Thus $d(a, y) > r$.

$$\therefore y \notin B[a, r].$$

Hence $y \in B[a, r]^c$.

$$\therefore B(x, r_1) \subseteq B[a, r]^c.$$

$\therefore B[a, r]^c$ is open in M.

$\therefore B[a, r]$ is closed in M.

Theorem 9. In any metric space M, **(i)** Φ is closed, **(ii)** M is closed.

Proof. Since $M^c = \Phi$ is open. M is open.

Similarly, $\Phi^c = M$ is open and hence is Φ is closed.

Note. We note that in any metric space M, Φ and M are both open and closed.

Theorem 10. In any metric space arbitrary intersection of closed sets is closed.

Proof. Let (M, d) be a metric space.

Let $\{A_i | i \in I\}$ be a collection of closed sets.

We claim that $\bigcap_{i \in I} A_i$ is closed.

We have $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} {A_i}^c$. (by De Morgan's law)

Since A_i is closed ${A_i}^c$ is open.

Hence $\bigcup_{i \in I} {A_i}^c$ is open. (By theorem 3)

$\therefore (\bigcap_{i \in I} A_i)^c$ is open.

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$\therefore \bigcap_{i\in I} A_i$ is closed.

Theorem 11. In any metric space the union of a finite number of closed set is closed.

Proof. Let (M, d) be a metric space.

Let $A_1, A_2, \dots, A_n$ be closed sets in M.

By De-Morgan's law $(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1{}^c \cap A_2{}^c \cap \dots \cap A_n{}^c$.
(by theorem 4)

Since each A_i is closed $A_i{}^c$ is open.

Hence $A_1{}^c \cap A_2{}^c \cap \dots \cap A_n{}^c$ is open.

$\therefore (A_1 \cup A_2 \cup \dots \cup A_n)^c$ is open.

Hence $A_1 \cup A_2 \cup \dots \cup A_n$ is closed.

Note. The union of an infinite collection of closed sets need not be closed. For example, consider $\mathbb{R}$ with usual metric.

Let $A_n = \left[\frac{1}{n}, 1\right]$ where $n = 1,2, \dots$.

Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \dots \dots$

$= (0,1]$ which is not closed in $\mathbb{R}$.

$\therefore \bigcup_{n=1}^{\infty} A_n$ is not closed.

Theorem 12. Let M be a metric space and M_1 be a subspace of M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 iff there exists a set F which is closed in M such that $F_1 = F \cap M_1$.

Proof. Let F_1 be closed in M_1.

$\therefore M_1 - F_1$ is closed in M_1.

$\therefore M_1 - F_1 = A \cap M_1$, where A is open in M. (by theorem 6)

Now, $F_1 = M_1 - (A \cap M_1)$.

$$= M_1 - A = A^c \cap M_1.$$

Also, since A is open in M, A^c is closed in M.

$\therefore F_1 = F \cap M_1$ where $F = A^c$ is closed in M.

Proof of the converse is similar.

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4.6 CLOSURE

Let (M, d) be a metric space. Let $A \subseteq M$. Consider the collection of all closed sets which contain A. This collection is non empty since at least M is a member of this collection.

Definition. Let A be a subset of metric space (M, d). The **closure** of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A.

Thus $\bar{A} = \bigcup\{B | B \text{ is closed in } M \text{ and } A \subseteq B\}$.

Note. Since intersection of any collection of closed set $\bar{A} \supseteq A$. Also if B is any closed set containing A then $\bar{A} \subseteq B$. Thus $\bar{A}$ is *the smallest closed set* containing A.

Theorem 13. A is closed iff $A = \bar{A}$.

Proof. Suppose $A = \bar{A}$.

Since $\bar{A}$ is closed A is closed.

Conversely, suppose A is closed. Then the smallest closed set containing A is A itself.

$$\therefore A = \bar{A}.$$

Note. In particular **(i)** $\Phi = \bar{\Phi}$ **(ii)** $\mathrm{M} = \bar{\mathrm{M}}$ **(iii)** $\bar{\bar{A}} = A$.

Example 1. Consider $\mathbb{R}$ with usual metric.

(a) Let $A = [0,1]$. We know that A is a closed set.

$$\therefore \bar{A} = A = [0,1].$$

(b) Let $A = (0,1)$. Then $[0,1]$ is a closed set containing $(0,1)$. Obviously $[0,1]$ is the smallest closed set containing $(0,1)$.

$$\therefore \bar{A} = [0,1].$$

Example 2. In a discrete metric space (M, d) any subset A of M is closed. Hence $\bar{A} = A$.

Theorem 14. Let (M, d) be a metric space. Let $A, B \subseteq M$.

Then (i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$.

(ii) $(\overline{A \cup B}) = \bar{A} \cup \bar{B}$.

(iii) $(\overline{A \cap B}) = \bar{A} \cap \bar{B}$.

Proof. (i) Let $A \subseteq B$.

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Now, $\bar{B} \supseteq B \supseteq A$.

$\therefore \bar{B}$ is a closed set containing A.

But $\bar{A}$ is the smallest closed set containing A.

$$\therefore \bar{A} \subseteq \bar{B}.$$

(ii)we have $A \subseteq A \cup B$.

$\therefore \bar{A} \subseteq \overline{A \cup B}$. (by (i)).

Similarly, $\bar{B} \subseteq \overline{A \cup B}$.
$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Now $\bar{A}$ is a closed set containing A and $\bar{B}$ is a closed set containing B.

$\therefore \bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$.

But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.

$$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$$

From (1) and (2) we get $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

(iii) We have $A \cap B \subseteq A$.

$\overline{A \cap B} \subseteq \bar{A}$. (by (i)).

Similar, $\overline{A \cap B} \subseteq \bar{B}$.

$$\therefore \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}.$$

Note. $\overline{A \cap B}$ need not be equal to $\bar{A} \cap \bar{B}$.

For example in $\mathbb{R}$ with usual metric, take $A = (0,1)$ and $= (1,2)$.

Then $A \cap B = \Phi$.

$$\therefore \overline{A \cap B} = \bar{\Phi} = \Phi.$$

But , $\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}$.

$$\therefore \overline{A \cap B} \neq \bar{A} \cap \bar{B}.$$

Note. In a metric space (M, d) if $E_1, E_2, \dots, E_n$ are subset of M then $\overline{E_1 \cup E_2 \cup \dots \cup E_{n.}} = \overline{E_1} \cup \overline{E_2} \dots \cup \overline{E_n}$. This is an extension of result (ii) of theorem 2.14.

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4.7 LIMIT POINT

In this section we introduce the concept of limit point of a set. This concept can be used to characterize closed sets and describe the closure of a set.

Definition. Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in M$. Then x is called a **limit poin**t or a **cluster point** or an **accumulation point** of A if every open ball with center x contains at least one point of A different from x.

(i.e.) $B(x,r) \cap (A - \{x\}) \neq \Phi$ for all $r > 0$.

The set of all limit points of A is called the **derived set** of A and is denoted by $D(A)$.

Note. x is not a limit point of A iff there exists an open ball $B(x,r)$ such that $B(x,r) \cap (A - \{x\}) = \Phi$.

Example 1. Consider $\mathbb{R}$ with usual metric.

(a) Let $A = [0,1]$.
Any open ball with center 0 is of the form $(-r,r)$ which contains a point of $[0,1)$ other that 0.
Hence 0 is a limit point of $[0,1)$.
Similarly 1 is a limit point of $[0,1)$.

2 is not a limit point of A, since

$$\left(2-\frac{1}{2}, 2+\frac{1}{2}\right) \cap [0,1) = \left(\frac{3}{2}, \frac{5}{2}\right) \cap [0,1) = \Phi.$$

In this case all points of $[0,1]$ are limit points of $[0,1)$ and no other points is a limit point.

Hence $D[0,1) = [0,1]$.

(b) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$. Here 0 is a limit point of A.

For, consider any open ball $(-r,r)$ with center 0.

Choose a positive integer n such that $\frac{1}{n} < r$.

Then $\frac{1}{n} \in (-r,r)$.

$\therefore$ $(-r,r)$ contains a point of A which is different from 0.

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$\therefore$ 0 is a limit point of A.

1 is not a limit point of A since

$$\left(1-\frac{1}{4},1+\frac{1}{4}\right)\cap(A-\{1\})=\left(\frac{3}{4},\frac{5}{4}\right)\cap\left\{\frac{1}{2},\frac{1}{3},\dots.,\frac{1}{n},\dots.\right\}=\Phi.$$

In fact any point except zero is not a limit point of A (verify).

$$\therefore D(A)=\{0\}.$$

(c) Consider $\mathbb{Q}$. Any real number x is a limit point of $\mathbb{Q}$, since any interval $(x+r,x-r)$ contains infinite number of rational numbers.

$$\therefore D(\mathbb{Q})=\mathbb{R}.$$

Example 2. In $\mathbb{R}\times\mathbb{R}$ with usual metric, $D(\mathbb{Q}\times\mathbb{Q})=\mathbb{R}\times\mathbb{R}$.

The proof is similar to example (d) of 1.

Example 3. Let (M,d) be a discrete metric space.

Let $A\subseteq M$. Let $x\in M$.

Then $B\left(x,\frac{1}{2}\right)\cap(A-\{x\})=\{x\}\cap(A-\{x\})=\Phi$.

$\therefore$ x is not a limit point of A.

Since $x\in M$ is arbitrary A has no limit point.

$$\therefore D(A)=\Phi.$$

Thus any subset of a discrete metric space has no limit point.

Example 4. Consider $\mathbb{C}$ with usual metric.

Let $A=\{z||z|<1\}$.

Then $D(A)=\{z|\,|z|\leq 1\}$.

Theorem 15. Let (M,d) be a metric space. Let $A\subseteq M$. Then x is a limit point of A iff each open ball with center x contains an infinite number of points of A.

Proof. Let x be a limit point of A.

Suppose an open ball $B(x,r)$ contains only a finite number of points of A.

Let $B(x,r)\cap(A-\{x\})=\{x_1,x_2,\dots.,x_n\}$.

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Let $r_1 = \min\{d(x, x_i) | i = 1,2, \dots, n\}$.

Since $x \neq x_i$, $d(x, x_i) > 0$ for all $i = 1,2, \dots, n$ and hence $r_1 > 0$.

Also $B(x, r_1) \cap (A - \{x\}) = \Phi$.

$\therefore$ x is not a limit point of A which is a contradiction.
Hence every open ball with center x contains infinite number of points of A.
The converse is obvious.

Corollary. Any finite subset of a metric space has no limit point.

Proof. Let A be a finite subset of M.

Suppose A has limit point say x. Then $B(x, r)$ contains infinite number of points of A. This is a contradiction since A is finite.

Theorem 16. Let M be a metric space and $A \subseteq M$. Then $\bar{A} = A \cup D(A)$.

Proof. Let $x \in A \cup D(A)$. We shall prove that $x \in \bar{A}$.

Suppose $x \notin \bar{A}$.

$\therefore$ $x \in M - \bar{A}$ and since $\bar{A}$ is closed $M - \bar{A}$ is open.

$\therefore$ There exists an open ball $B(x, r) \subseteq M - \bar{A}$.

$$\therefore B(x, r) \cap \bar{A} = \Phi.$$

$\therefore$ $B(x, r) \cap A = \Phi$. (since $A \subseteq \bar{A}$)

$\therefore$ $x \notin A \cup D(A)$ which is a contradiction.

$$\therefore x \in \bar{A}.$$

$\therefore$ $A \cup D(A) \subseteq \bar{A}$. .....(1)

Now let $x \in \bar{A}$. To prove $x \in A \cup D(A)$.

If $x \in A$ clearly $x \in A \cup D(A)$.

Suppose $x \notin A$. We claim that $x \in D(A)$.

Suppose $x \notin D(A)$. Then there exists an open ball $B(x, r)$ such that $B(x, r) \cap A = \Phi$.

$\therefore$ $B(x, r)^c \supseteq A$ and $B(x, r)^c$ is closed.

But $\bar{A}$ is the smallest closed set containing A.

$$\therefore \bar{A} \subseteq B(x, r)^c.$$

But $x \in \bar{A}$ and $x \notin B(x,r)^c$ which is a contradiction.

Hence $x \in D(A)$.

$$\therefore x \in A \cup D(A).$$

$\therefore \bar{A} \subseteq A \cup D(A)$. ....(2)

From (1) and (2) we get $\bar{A} = A \cup D(A)$.

Corollary 1. A is closed iff A contains all its limit points.

i.e. A is closed iff $D(A) \subseteq A$.

Proof. A is closed $\Leftrightarrow A = \bar{A}$.

$$\Leftrightarrow A = A \cup D(A)$$

$$\Leftrightarrow D(A) \subseteq A.$$

Corollary 2. $x \in \bar{A} \Leftrightarrow B(x,r) \cap A \neq \Phi$ for all $r > 0$.

Proof. Let $x \in \bar{A}$, then $x \in A \cup D(A)$.

$\therefore x \in A$ or $x \in D(A)$.

If $x \in A$ then $x \in B(x,r) \cap A$.

If $x \in D(A)$ then $B(x,r) \cap A \neq \Phi$ for all $r > 0$.

Hence in both cases $B(x,r) \cap A \neq \Phi$ for all $r > 0$.

Conversely, suppose $B(x,r) \cap A \neq \Phi$ for all $r > 0$.

We have to prove that $x \in \bar{A}$.

If $x \in A$ trivially $x \in \bar{A}$.

Let $x \notin A$. Then $A - \{x\} = A$.

$\therefore B(x,r) \cap (A - \{x\}) \neq \Phi$.

$\therefore x \in D(A)$.

$\therefore x \in \bar{A}$.

Corollary 3. $x \in \bar{A} \Leftrightarrow G \cap A \neq \Phi$ for every open set G containing x.

Proof. Let $x \in \bar{A}$.

Let G be an open set containing x. Then there exists $r > 0$ such that $B(x,r) \subseteq G$.

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Also, since $x \in \bar{A}$, $B(x,r) \cap A \neq \Phi$.

$$\therefore G \cap A \neq \Phi.$$

Conversely, suppose $G \cap A \neq \Phi$ suppose $G \cap A \neq \Phi$ for every open set G

containing x.

Since $B(x,r)$ is an open set containing x, we have $B(x,r) \cap A \neq \Phi$.

$$\therefore x \in \bar{A}.$$

Example 1. Consider $\mathbb{R}$ with usual metric.

(a) Let $A = [0,1)$.

Then $\bar{A} = A \cup D(A)$.

$= [0,1) \cup [0,1]$.

$= [0,1]$.

(b) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$

Then $\bar{A} = A \cup D(A)$.

$$= \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\} \cup \{0\}.$$

(c) $\bar{\mathbb{Z}} = \mathbb{Z} \cup D(\mathbb{Z})$.

$$= \mathbb{Z} \cup \Phi = \mathbb{Z}.$$

$\therefore$ $\mathbb{Z}$ is closed.

(d) $\bar{\mathbb{Q}} = \mathbb{Q} \cup D(\mathbb{Q})$.

$= \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

$\therefore$ $\mathbb{Q}$ is not closed.

Example 2. In $\mathbb{R} \times \mathbb{R}$ with usual metric.

$$\overline{\mathbb{Q} \times \mathbb{Q}} = (\mathbb{Q} \times \mathbb{Q}) \cup D(\mathbb{Q} \times \mathbb{Q}).$$

$= (\mathbb{Q} \times \mathbb{Q}) \cup (\mathbb{R} \times \mathbb{R})$.

$$= \mathbb{R} \times \mathbb{R}.$$

$\therefore$ $\mathbb{Q} \times \mathbb{Q}$ is not closed.

SOLVED PROBLEM

Problem 1. Prove that for any subset A of a metric space, $d(A) = d(\bar{A})$ where $d(A)$ is the diameter of A.

Solution. We have $A \subseteq \bar{A}$.

$\therefore$ $d(A) \leq d(\bar{A})$ .....(1)

Now, let $\varepsilon > 0$ be given. We claim that $d(\bar{A}) \leq \mathrm{d(A)} + \varepsilon$.

Let $x, y \in \bar{A}$.

$\therefore B\left(x, \frac{1}{2}\varepsilon\right) \cap A \neq \Phi$ and $B\left(y, \frac{1}{2}\varepsilon\right) \cap A \neq \Phi$ (by cor. 2)

Let $x_1 \in B\left(x, \frac{1}{2}\varepsilon\right) \cap A$ and $x_2 \in B\left(y, \frac{1}{2}\varepsilon\right) \cap A$.

$\therefore x_1 \in B\left(x, \frac{1}{2}\varepsilon\right)$ and $x_2 \in B\left(y, \frac{1}{2}\varepsilon\right)$.

$\therefore d(x, x_1) < \frac{1}{2}\varepsilon$ and $d(y, x_2) < \frac{1}{2}\varepsilon$. .....(2)

Also, $x_1 \in A$ and $x_2 \in A \Rightarrow d(x_1, x_2) \leq d(A)$. .....(3)

Now, $d(x, y) \leq d(x, x_1) + d(x_1, x_2) + d(y, x_2)$.

$< \frac{1}{2}\varepsilon + d(A) + \frac{1}{2}\varepsilon$. (by (2) and (3))

$= d(A) + \varepsilon$.

Thus $d(x, y) < d(A) + \varepsilon$.

$\therefore l.u.b.\{d(x, y) | x, y \in \bar{A}\} \leq d(A) + \varepsilon$.

Now, since ε is arbitrary, we have $d(\bar{A}) \leq d(A)$. ....(4)

By (1) and (4), we get $d(A) = d(\bar{A})$.

4.8 DENSE SETS

Definition. A subset A of a metric space M is said to be **dense** in M or everywhere dense if $\bar{A} = M$.

Definition. A metric space M is said to be separable if there exists a countable dense subset in M.

Example 1. Let M be a metric space. Trivially, M is dense in M.

Hence any countable metric space is separable.

Example 2. In $\mathbb{R}$ with usual metric $\mathbb{Q}$ is dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$.

Further $\mathbb{Q}$ is countable.

Hence $\mathbb{R}$ is separable.

Example 3. Let M be a discrete metric space.

Let $A \subset M$ and $A \neq M$.

Since A is closed, $\bar{A} = A$.

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$\therefore A$ is not dense.

Hence any uncountable discrete metric space is not separable.

Example 4. In $\mathbb{R} \times \mathbb{R}$ with usual metric $\mathbb{Q} \times \mathbb{Q}$ is a dense set, since $\overline{\mathbb{Q} \times \mathbb{Q}} = \mathbb{R} \times \mathbb{R}$.

Also $\mathbb{Q}$ is countable and hence $\mathbb{Q} \times \mathbb{Q}$ is countable.

$\therefore \mathbb{R} \times \mathbb{R}$ is separable.

Theorem 18. Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

(i) A is dense in M.
(ii) The only open set disjoint from A is M.
(iii) The only open set disjoint from A is Φ.
(iv) A intersections every non-empty open-set.
(v) A intersections every open ball.

Proof.

(i)⇒(ii).

Suppose A is dense in M.

Then $\bar{A} = M$.

Now, let $F \subseteq M$ be any closed set containing A.

Since $\bar{A}$ is the smallest closed set containing A, we have $\bar{A} \subseteq F$.

Hence $M \subseteq F$. (by (1)).

$$\therefore M = F.$$

$\therefore$ The only closed set which contains A is M.

(ii)⇒(iii). Suppose (iii) is not true.

Then there exists a non-empty open set B such that $B \cap A = \Phi$.

$\therefore B^c$ is a closed set and $B^c \supseteq A$.

Further, since $B \neq \Phi$ we have $B^c \neq$ M which is a contradiction to (ii).

Hence **(ii)⇒(iii).**

Obviously **(iii)⇒(iv).**

(iv)⇒(v), since every open ball $B(x, r)$ intersect A.

Then by corollary (2) of theorem 16, $x \in \bar{A}$.

$\therefore M \subseteq \bar{A}$.

But trivially $\bar{A} \subseteq M$.

$\therefore \bar{A} = M$.

$\therefore A$ is dense in M.

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SOLVED PROBLEM

Problem 1. Give an example of a set E such that both E and E^c are dense in $\mathbb{R}$.

Solution. Let $E = \mathbb{Q}$.

Since any open ball $B(x,r) = (x-r, x+r)$ contains both irrational $\mathbb{Q}$ and $\mathbb{Q}^c$.

Hence $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$. (by theorem 17)

CHECK YOUR PROGRESS

1. Show that $[1,2] \cup [3,4]$ is open in $\mathbb{R}$.
2. When did the set of interior point of A is equal to the set A?
3. Is $\{0\}$ is open or not?
4. Show that the set of irrational numbers is not open in $\mathbb{R}$.
5. Show that every subset of a discrete metric space is closed.
6. Show that $\mathbb{Z}$ has no limit point.

4.9 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Let $M = \mathbb{R}$ and $M_1 = [1,2] \cup [3,4]$. Let $A_1 = [1,2]$. Then $A_1 = [1,2] = \left(\frac{1}{2}, \frac{5}{2}\right) \cap M_1$. Therefore, $(1,2)$ is open in M_1. Similarly $[3,4]$ is open in M_1.
2. In a discrete metric space M, $Int\ A = A$ for any subset A of M.
3. In $\mathbb{R}$ with usual metric the set $\{0\}$ is not an open set since, any open ball with center 0 is not contained in $\{0\}$.
4. Proof is similar to that of example 7.
5. Let (M,d) be a discrete metric space. Let $A \subseteq M$. Since every subset of a discrete metric space is open A^c is open. Therefore, A is closed.
6. Let x is an integer, then $B\left(x, \frac{1}{2}\right) = \left(x - \frac{1}{2}, x + \frac{1}{2}\right)$ does not contain any integer other that x. Hence x is not a limit point of $\mathbb{Z}$. If x is not an integer, let n be the integer which is closest to

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x. Choose r such that $0 < r < |x - n|$. Then $B(x,r) = (x - r, x + r)$ contains no integer. Hence x is not a limit point of $\mathbb{Z}$. Since x is arbitrary $\mathbb{Z}$ has no limit point. Therefore, $D(\mathbb{Z}) = \Phi$.

4.10 SUMMARY

1. Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with the same metric d. we say that (M_1, d) is a **subspace** of (M, d).

2. Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an **interior** of A if there exists a positive real number r such that $B(x, r) \subseteq A$.

3. The set of all interior points of A is called the interior of A and it is denoted by $\boldsymbol{Int\ A}$.

4. A is open iff A=Int A. In particular $Int\ \Phi = \Phi$ and $Int\ \mathrm{M} = \mathrm{M}$.

5. Let (M, d) be a metric space. Let A be a subet of M. Then A is said to be **open** in M if for every $x \in A$ there exists a positive real number r such that $B(x, r) \subseteq A$.

6. In any metric space (M, d) each open ball is an open set.

7. In any metric space the union of any family of open sets is open.

8. In any metric space the intersection of a finite number of open sets is open.

9. Prove that any open subset of R can be expressed as the union of a countable number of mutually disjoint open intervals.

10. In any metric space every closed ball is a closed set.

11. In any metric space M, **(i)** Φ is open, **(ii)** M is open.

12. In any metric space M, **(i)** Φ is closed, **(ii)** M is closed.

13. In any metric space arbitrary intersection of closed sets is closed.

4.11 KEYWORDS

1. **Subspaces:** Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with the same metric d. we say that (M_1, d) is a subspace of (M, d).

2. **Interior:** Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an interior of A if there exists a positive real number r such that $B(x, r) \subseteq A$.

3. $\boldsymbol{Int\ A}$**:** The set of all interior points of A is called the interior of A and it is denoted by $Int\ A$.

4. **Open:** Let (M, d) be a metric space. Let A be a subet of M. Then A is said to be open in M if for every $x \in A$ there exists a positive real number r such that $B(x, r) \subseteq A$.

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5. **Equivalent:** Let d and ρ be the two metrics on M. Then the metrics d and ρ are said to be equivalent if the open sets of (M,ρ) are the open sets of (M,d) and conversely.
6. **Closed:** Let (M,d) be a metric space. Let $A \subseteq M$. Then A is said to be closed in M if the complement of A is open in M.
7. **Closed ball or closed sphere:** Let (M,d) be a metric space. Let $a \in M$. Let r be any positive real number. Then the closed ball or the closed sphere with center a and radius r, denoted by $B_d[a,r]$, is defined by $B_d[a,r] = \{x \in M | d(a,x) \leq r\}$. When the metric d under consideration is clear we write $B[a,r]$ instead of $B_d[a,r]$.
8. **Closure:** Let A be a subset of metric space (M,d). The closure of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A.
9. **Limit:** Let (M,d) be a metric space. Let $A \subseteq M$. Let $x \in M$. Then x is called a limit point or a cluster point or an accumulation point of A if every open ball with center x contains at least one point of A different from x.
10. **Derived set:** The set of all limit points of A is called the derived set of A and is denoted by $D(A)$.
11. **Dense:** A subset A of a metric space M is said to be dense in M or everywhere dense if $\bar{A} = M$.

4.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Given an example of a metric space M and a non-empty proper subspace $M_1$of M such that every open set in M_1 is also an open set in M.
2. Determine the interior of $\mathbb{Z}$ which is the subsets of $\mathbb{R}$.
3. Prove that any finite subset of a metric space is closed.
4. Given an example to show that in a metric space closure of an open ball $B(x,r)$ need not be equal to the corresponding closed ball $B[x,r]$.
5. Prove that the set of all limit points of a subset of a metric space is closed.
6. Prove that any open ball is a non-empty open set.
7. Prove that $\mathbb{R}^n$ with usual metric is separable.
8. With usual metric show that $\mathbb{Q}$ is dense in $\mathbb{R}$.
9. Prove that in a discrete metric space every set is both open and closed.
10. Show that a set which is not closed is open.

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4.13 FURTHER READINGS

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BLOCK- II
CONTINUITY AND POWER SERIES

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UNIT-5 COMPLETE METRIC SPACES

Structure

5.0 INTRODUCTION

The reader is familiar with the concept of convergent sequences and Cauchy sequences in $\mathbb{R}$. In this chapter we generalize these concept to sequence in any metric space.

5.1 OBJECTIVE

After going through this unit, you will be able to:

- Understand what is meant by complete.
- Determine converges of a sequence and Cauchy sequence.
- Discuss Baire's Category theorem.

5.2 COMPLETENESS

Definition. Let (M, d) be a metric space. Let $(x_n) = x_1, x_2, \ldots, x_n, \ldots$ be a sequence of point in M. Let $x \in M$. We say (x_n) is **converges** to x if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ such that for all $n \geq n_0$. Also x is called a **limit** of (x_n).

If (x_n) converges to x we write $\lim_{n\to\infty} x_n = x$ or $(x_n) \to x$.

Note 1. $(x_n) \to x$ iff for each open ball $B(x, \varepsilon)$ with center x there exists a positive integer n_0 such that $x_n \in B(x, \varepsilon)$ for all $n \geq n_0$.

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Thus the open ball $B(x,\varepsilon)$ contains all but a finite number of terms of the sequence.

Note 2. $(x_n) \to x$ iff the sequence of real numbers $(d(x_n,x)) \to 0$.

Theorem 1. For a convergence sequence (x_n) the limit is unique.

Proof. Suppose $(x_n) \to x$ and $(x_n) \to y$.

Let $\varepsilon > 0$ be given. Then there exist positive integersn_1 and n_2 such that $d(x_n,x) < \frac{1}{2}\varepsilon$ for all $n \geq n_1$ and $d(x_n,y) < \frac{1}{2}\varepsilon$ for all $n \geq n_2$.

Let m be a positive integer such that $m \geq n_1, n_2$.

Then $d(x,y) \leq d(x,x_m) + d(x_m,y)$.

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

$\therefore d(x,y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary $d(x,y) = 0$.

$\therefore d(x,y) = 0$.

$\therefore x = y$.

Note. In view of the above theorem if $(x_n) \to x$ then x is called the limit of the sequence (x_n).

The connection between the limit of a sequence and limit of a sequence and limit point of a set is given in the following theorem.

Theorem 2. Let M be a metric space and $A \subseteq M$. Then

(i) $x \in \bar{A}$ iff there exists a sequence (x_n) of distinct points of A such that $(x_n) \to x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

Proof. Let $x \in \bar{A}$.

Then $x \in A \cup D(A)$. (by theorem 16 in unit 4)

$\therefore$ $x \in A$ and $x \in D(A)$

If $x \in A$, then the constant sequence $x, x, \ldots$ is a sequence in A converging to x.

If $x \in D(A)$ then the open ball $B(x, \frac{1}{n})$ contains infinite number of points of A. (by theorem 15 of unit 4)

$\therefore$ We can choose $x_n \in B(x, \frac{1}{n}) \cap A$ such that $x_n \neq x_1, x_2, \ldots, x_{n-1}$ for each n.

$\therefore (x_n)$ be a sequence of distinct points in$A$.

Also $d(x_n, x) < \frac{1}{n}$ for all n.

$\lim_{n\to\infty} d(x_n, x) = 0$.

$\therefore (x_n) \to x$.

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \to x$.

Then for any $r > 0$ there exists a positive integer n_0 such that $d(x_n, x) < r$ for all $n \geq n_0$.

$\therefore B(x, r) \cap A \neq \Phi$.

$\therefore\ x \in \bar{A}$. (by corollary 2 of theorem 16 in unit 4)

Further if (x_n) is a sequence of distinct points, $B(x, r) \cap A$ is infinite.

$\therefore\ x \in D(A)$.

$\therefore\ x$ is a limit point of A.

Definition. Let (M, d) be a metric space. Let (x_n) be a sequence of points in M. (x_n) is said to be a **Cauchy sequence** in M if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_{0.}$

Theorem 3. Let (M, d) be a metric space. Then any convergence sequence in M is a Cauchy sequence.

Proof. Let (x_n) be a convergent sequence in M converging to $x \in M$.

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_n, x) < \frac{1}{2}\varepsilon$ for all $n \geq n_{0.}$

$d(x_m, x_n) \leq d(x_m, x) + d(x, x_m)$.

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$ for all $m, n \geq n_{0.}$

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Thus $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_{0.}$

$\therefore (x_n)$ is a Cauchy sequence.

Note. The converse of the above theorem is not true.

For example, consider the metric space (0,1] with usual metric.

$(\frac{1}{n})$ is a Cauchy sequence in (0,1].

But this sequence does not converge to any point.

Definition. A metric space M is said to be **complete** if every Cauchy sequence in M converges to a point in M.

Example1. $\mathbb{R}$ with usual metric is complete. This is a fundamental fact of elementary analysis and a proof of this fact is given is unit 13

Note. The metric space (0,1] with usual metric is not complete (refer note given above)

Example 2. $\mathbb{C}$ with usual metric is complete.

Proof. Let (z_n) be a Cauchy sequence in $\mathbb{C}$.

Let $z_n = x_n + iy_n$ where $x_n, y_n \in \mathbb{R}$.

We claim that (x_n) and (y_n) are Cauchy sequence in $\mathbb{R}$.

Let $\varepsilon > 0$ be given.

Since (z_n) is a Cauchy sequence, there exists a positive integer n_0 such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq n_0$.

Now, $|x_n - x_m| \leq |z_n - z_m|$ and $|y_n - y_m| \leq |z_n - z_m|$.

Hence $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$ and $|y_n - y_m| < \varepsilon$ for all $n, m \geq n_0$.

$\therefore (x_n)$ and (y_n) are Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, there exists $x, y \in \mathbb{R}$ such that $(x_n) \to x$ and $(y_n) \to y$.

Let $z = x + iy$. We claim that $(z_n) \to z$.

We have $|z_n - z| = |(x_n + iy_n) - (x + iy)|$

$= |(x_n - x) + i(y_n - y)|$

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Now, let $\varepsilon > 0$ be given.

Since $(x_n) \to x$ and $(y_n) \to y$ there exist positive integer n_1 and n_2 such that $|x_n - x| < \frac{1}{2}\varepsilon$ for all $n \geq n_1$ and $|y_n - y| < \frac{1}{2}\varepsilon$ for all $n \geq n_2$.

Let $n_3 = \max\{n_1, n_2\}$.

From (1) we get $|z_n - z| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for all $n \geq n_3$.

$\therefore (z_n) \to z$.

$\therefore \mathbb{C}$ is complete.

Example 3. Any discrete metric space is complete.

Proof. Let (M, d) be a discrete metric space.

Let (x_n) be a Cauchy sequence in M.

Then there exists a positive integer n_0 such that $d(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$.

Since d is the discrete metric distance between any two points is either 0 or 1.

$\therefore d(x_n, x_m) = 0$ for all $n, m \geq n_0$.

$\therefore x_n = x_{n_0} = x$ (say) $n \geq n_0$.

$\therefore d(x_n, x) = 0$ for all $n \geq n_0$.

$\therefore (x_n) \to x$. Hence M is complete.

Example 4. $\mathbb{R}^n$ with usual metric is complete.

Proof. Let $(x_p) = (x_{p_1}, \ldots., x_{p_n})$. Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \left[\sum_{k=1}^{n}(x_{p_k} - x_{q_k})^2\right]^{1/2} < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \sum_{k=1}^{n}(x_{p_k} - x_{q_k})^2 < \varepsilon^2$ for all $p, q \geq n_0$.

$\therefore$ For each $k = 1,2, \ldots n$ we have

$|x_{p_k} - x_{q_k}| < \varepsilon$ for all $p, q \geq n_0$.

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$\therefore (x_{p_k})$ is a Cauchy sequence in $\mathbb{R}$ for each $k = 1,2, \dots, n$.

Since $\mathbb{R}$ is complete, there exists $y_k \in \mathbb{R}$ such that $(x_{p_k}) \to y_k$.

Let $y = (y_1, y_2, \dots, y_n)$. We claim that $(x_p) \to y$.

Since $(x_{p_k}) \to y_k$ there exists a positive integer m_k such that $|x_{p_k} - y_k| < \frac{\varepsilon}{\sqrt{n}}$ for all $p \geq m_k$.

Let $m_0 = \max\{m_1, \dots, m_n\}$.

Then $d(x_p, y) = \left[\sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2\right]^{1/2}$

$< [n\left(\frac{\varepsilon}{\sqrt{n}}\right)^2]^{1/2}$ for all $p \geq m_0$.

$= \varepsilon$ for all $p \geq m_0$.

Thus $d(x_p, y) < \varepsilon$ for all $p \geq m_0$.

$\therefore (x_p) \to y$. Hence $\mathbb{R}^n$ is complete.

Example 5. l_2 is complete.

Proof. Let (x_p) be a Cauchy sequence in l_2.

Let $(x_p) = (x_{p_1}, \dots., x_{p_n})$.

Let $\varepsilon > 0$ be given. Then there exists a positive integer n_0 such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geq n_0$.

$(i.e.) \left[\sum_{n=1}^{\infty} (x_{p_n} - x_{q_n})^2\right]^{1/2} < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \sum_{n=1}^{\infty} (x_{p_n} - x_{q_n})^2 < \varepsilon^2$ for all $p, q \geq n_0$. ....(1)

For each $n = 1,2, \dots..$ we have

$|x_{p_n} - x_{q_n}| < \varepsilon$ for all $p, q \geq n_0$.

$\therefore (x_{p_n})$ is a Cauchy sequence in $\mathbb{R}$ for each n.

Since $\mathbb{R}$ is complete, there exists $y_n \in \mathbb{R}$ such that

$(x_{p_n}) \to y_n$. ......(2)

Let $y = (y_1, y_2, \dots, y_n, \dots.)$.

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We claim that $y \in l_2$ and $(x_p) \to y$.

For any fixed positive integer m, we have

Let $\varepsilon > 0$ be given. Then there exists a positive integer n_0 such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geq n_0$.

$\sum_{n=1}^{m}(x_{p_n} - x_{q_n})^2 < \varepsilon^2$ for all $p, q \geq n_0$. (using (1))

Fixing q and allowing $p \to \infty$ in this finite sum we get

$\sum_{n=1}^{m}(y_n - x_{q_n})^2 \leq \varepsilon^2$ for all $q \geq n_0$. (using (2))

Since this is true for every positive integer m

$\sum_{n=1}^{\infty}(y_n - x_{q_n})^2 \leq \varepsilon^2$ for all $q \geq n_0$. ....(3)

Now, $[\sum_{n=1}^{\infty}|y_n|^2]^{1/2} = [\sum_{n=1}^{\infty}|y_n - x_{q_n} + x_{q_n}|^2]^{1/2}$

$\leq [\sum_{n=1}^{\infty}|y_n - x_{q_n}|^2]^{1/2} + [\sum_{n=1}^{\infty}|x_{q_n}|^2]^{1/2}$ (by Minkowski's inequality)

$\leq \varepsilon + [\sum_{n=1}^{\infty}|x_{q_n}|^2]^{1/2}$ for all $q \geq n_0$. (using (3))

Since $x_q \in l_2$ we have $[\sum_{n=1}^{\infty}|x_{q_n}|^2]^{1/2}$ converges.

$\therefore [\sum_{n=1}^{\infty}|y_n|^2]^{1/2}$ converges.

$\therefore y \in l_2$.

Also (3) gives $d(y, x_p) \leq \varepsilon$ for all $p \geq n_0$.

$\therefore (x_p) \to y$.

Hence l_2 is complete.

Note. A subspace of a complete metric space need not be complete.

For example $\mathbb{R}$ with usual metric is complete. But the subspace $(0,1]$ is not complete. (refer example 1).

In the next theorem we give a necessary and sufficient condition for a subspace of a complete metric space to be complete.

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Theorem 4. A subset A of a complete metric space M is complete iff A is closed.

Proof. Suppose A is complete.

To prove that A is closed, we shall prove that A contains all its limit points.

Let x be a limit point of A.

Then by theorem 2, there exists a sequence (x_n) in A such that $(x_n) \to x$.

Since A is complete $x \in A$.

$\therefore$ A contains all its limit points.

Hence A is closed.

Conversely, let A be a closed subset of M.

Let (x_n) be a Cauchy sequence in A.

Then (x_n) be a Cauchy sequence in M also and since M is complete there exists $x \in M$ such that $(x_n) \to x$. Thus (x_n) is a sequence in A converging to x.

$\therefore x \in \bar{A}$. (by theorem 2)

Now, since A is closed $A = \bar{A}$.

$\therefore x \in A$.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

$\therefore$ A is complete.

Note 1. [0,1] with usual metric is complete since it is a closed subset of the complete metric space $\mathbb{R}$.

Note 2. Consider $\mathbb{Q}$. Since $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}$ is not a closed subset of $\mathbb{R}$. Hence $\mathbb{Q}$ is not complete.

Solved problems

Problem 1. Let A, B be subsets of $\mathbb{R}$. Prove that $\overline{A \times B} = \bar{A} \times \bar{B}$.

Solution. Let $(x, y) \in \overline{A \times B}$.

$\therefore$ There exists a sequence $((x_n, y_n)) \in A \times B$ such that $((x_n, y_n)) \to (x, y)$. (by theorem 2)

NOTES

$\therefore (x_n) \to (x)$ and $(y_n) \to (y)$.

Also, (x_n) is a sequence in A and (y_n) is a sequence in B.

$\therefore x \in \bar{A}$ and $y \in \bar{B}$. (by theorem 2)

$\therefore (x, y) \in \bar{A} \times \bar{B}$.

$\therefore \overline{A \times B} \subseteq \bar{A} \times \bar{B}$. .....(1)

Now, let $(x, y) \in \bar{A} \times \bar{B}$.

$\therefore x \in \bar{A}$ and $y \in \bar{B}$.

$\therefore$ There exists a sequence (x_n) in A and a sequence (y_n) in B such that $(x_n) \to (x)$ and $(y_n) \to (y)$.

$\therefore \big((x_n, y_n)\big)$ is a sequence in $A \times B$ which converges to (x, y).

$\therefore (x, y) \in \overline{A \times B}$.

$\therefore \bar{A} \times \bar{B} \subseteq \overline{A \times B}$. ......(2)

$\therefore$ By (1) and (2) we get $\overline{A \times B} = \bar{A} \times \bar{B}$.

Theorem 5. (Cantor's Intersection Theorem)

Let M be a metric space. M is complete iff for every sequence (F_n) of non-empty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ and $(d(F_n)) \to 0$. $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Proof. Let M be a complete metric space.

Let (F_n) be a sequence of closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ ...(1)

And $(d(F_n)) \to 0$. ....(2)

We claim that $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

For each positive integer n, choose a point $x_n \in F_n$.

By (1), $x_n, x_{n+1}, x_{n+2}, \ldots$ all lie in F_n.

(i.e.) $x_m \in F_n$ for all $m \geq n$. ....(3)

NOTES

Since $d((F_n)) \to 0$, given $\varepsilon > 0$, there exists a positive integer n_0, such that $d(F_n) < \varepsilon$ for all $n \geq n_0$.

In particular, $d(F_n) < \varepsilon$. .....(4)

$\therefore d(x, y) < \varepsilon$ for all $x, y \in F_n$.

Now, $x_m \in F_{n_0}$ for all $m \geq n_0$. (by (3))

$\therefore m, n \geq n_0 \Rightarrow x_m, x_n \in F_{n_0}$.

$d(x_m, x_n) < \varepsilon$. (by (4))

$\therefore (x_n)$ is a Cauchy sequence in M.

Since M is complete there exists a point $x \in M$ such that $(x_n) \to x$.

We claim that $x \in \bigcap_{n=1} F_n$.

Now, for any positive integer n, $x_n, x_{n+1}, x_{n+2}, \ldots$ is a sequence in F_n and this sequence converges to x.

$\therefore x \in \overline{F_n}$. (by theorem 2)

But $\overline{F_n}$ is closed and hence $\overline{F_n} = F_n$.

$\therefore x \in F_n$.

$\therefore x \in \bigcap_{n=1}^{\infty} F_n$.

Hence $\bigcap_{n=1}^{\infty} F_n \neq \Phi$.

To prove the converse let, (x_n) be any Cauchy sequence in M.

let $F_1 = \{x_1, x_2, \ldots, x_n, \ldots\}$.

$F_2 = \{x_2, x_3, \ldots, x_n, \ldots\}$.

… … … …. …. ….

… … … …. …. ….

… … … …. …. ….

$F_n = \{x_n, x_{n+1}, \ldots \ldots\}$.

Clearly $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$

$\therefore \overline{F_1} \supseteq \overline{F_2} \supseteq \cdots \supseteq \overline{F_n} \supseteq \cdots$

$\therefore (\overline{F_1})$ is a decreasing sequence of closed sets.

NOTES

Now, since (x_n) is a Cauchy sequence, given $\varepsilon > 0$ there exists a positive integer n_0, such that $d(x_m, x_n) < \varepsilon$ for all $n, m \geq n_0$.

$\therefore$ For any integer $n \geq n_0$, the distance between any two points of F_n is less than ε.

$\therefore d(F_n) < \varepsilon$ for all $n \geq n_0$.

But $d(F_n) = d(\overline{F_n})$.

$\therefore d(\overline{F_n}) < \varepsilon$ for all $n \geq n_0$. ......(5)

$\therefore (d(\overline{F_n})) \to 0$.

Hence $\bigcap_{n=1}^{\infty} \overline{F_n} \neq \Phi$.

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$. Then x and $x_n \in \overline{F_n}$.

$\therefore d(x_n, x) < d(\overline{F_n})$.

$\therefore d(x_n, x) < \varepsilon$ for all $n \geq n_0$. (by (5))

$\therefore (x_n) \to x$.

$\therefore M$ is complete.

Note 1. In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

For, suppose that $\bigcap_{n=1}^{\infty} F_n$ contains two distinct points x and y.

Then $d(F_n) \geq d(x, y)$ for all n.

$\therefore (d(F_n))$ does not tend to zero which is a contradiction.

$\therefore \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Note 2. In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if each F_n is not closed.

For example, consider $F_n = (0, \frac{1}{n})$ in $\mathbb{R}$.

Clearly $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ and $\big(d(F_n)\big) = \left(\frac{1}{n}\right) \to 0$ as $n \to \infty$.

But $\bigcap_{n=1}^{\infty} F_n = \Phi$.

Note 3. In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if the hypothesis $(d(F_n)) \to 0$ is omitted.

For example, consider $F_n = [n, \infty)$ in $\mathbb{R}$.

NOTES

Clearly (F_n) is a sequence of closed sets and $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$.

Also $\bigcap_{n=1}^{\infty} F_n = \Phi$.

Here, $d(F_n) = \infty$ for all n and hence the hypothesis $(d(F_n)) \to 0$ is not true.

5.3 BAIRE'S CATEGORY THEOREM

In this section we prove a fundamental property of complete metric space called Baire's Category theorem.

Definition. A subset A of a metric space M is said to be **nowhere dense** in M if $Int\ \bar{A} = \Phi$.

Definition. A subset A of a metric space M is said to be of **first category** in M if A can be expressed as a countable union of nowhere dense sets.

A set which is not of first category is of **second category.**

Note. If A is of first category then $A = \bigcup_{n=1}^{\infty} E_n$ where E_n is nowhere dense subsets in M.

Example 1. In $\mathbb{R}$ with usual metric $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots.\}$ is nowhere dense.

For, $\bar{A} = A \cup D(A) = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots.\}$.

Clearly, $Int\ \bar{A} = \Phi$.

Example 2. In any discrete metric space M, any non-empty subset A is not nowhere dense.

For, in a discrete metric space every subset is both open and closed.

$\therefore\ \bar{A} = Int\ \bar{A} = Int\ A = A$.

$\therefore Int\ \bar{A} \neq \Phi$.

$\therefore A$ is not nowhere dense.

Example 3. In $\mathbb{R}$ with usual metric any finite subset A is nowhere dense. For, let A be any finite subset of $\mathbb{R}$.

Then A is closed and hence $A = \bar{A}$.

Also since A is finite, no point of A is an interior point of A.

$\therefore Int\ \bar{A} = Int\ A = \Phi$.

$\therefore A$ is nowhere dense.

Note. If A and B are sets of first category in a metric space M then $A \cup B$ is also of first category.

For, since A and B are of first category in M we have $A = \bigcup_{n=1}^{\infty} E_n$ and $B = \bigcup_{n=1}^{\infty} H_n$ where E_n and H_n are nowhere dense subsets in M.

$\therefore A \cup B$ is a countable union of nowhere dense subsets of M. (refer theorem 7 of unit 1)

Hence $A \cup B$ is of first category.

We now give equivalent characterizations for nowhere dense sets.

Theorem 6. Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

(i) A is nowhere dense in M.

(ii) $\bar{A}$ does not contain any non-empty open set.

(iii) Each non-empty open set has a non-empty open subset disjoint from $\bar{A}$.

(iv) Each non-empty open set has a non-empty open subset disjoint from A.

(v) Each non-empty open set contains an open sphere disjoint from A.

Proof is left as an exercise to the reader.

Theorem 7. (Baire's Category Theorem)

Any complete metric space is of second category.

Proof.

Let M be a complete metric space.

We claim that M is not of first category.

Let (A_n) be a sequence of nowhere dense sets in M.

We claim that $\bigcup_{n=1}^{\infty} A_n \neq M$.

Since M is open and A_1 is nowhere dense, there exists an open ball say B_1 of radius less that 1 such that B_1 is disjoint from A_1. (refer theorem 3.6)

NOTES

Let F_1 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now $Int\ F_1$ is open and A_2 is nowhere dense.

$\therefore$ $Int\ F_1$ contains an open ball B_2 of radius less than $\frac{1}{2}$ such that B_2 is disjoint from A_2.

Let F_2 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2. Now $Int\ F_2$ is open and A_2 is nowhere dense.

$\therefore$ $Int\ F_2$ contains an open ball B_3 of radius less than $\frac{1}{4}$ such that B_2 is disjoint from A_3.

Let F_3 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_3.

Proceeding like this we get a sequence of non-empty closed balls exists a point x in$M$ such that

$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ .and $d(F_n) < \frac{1}{2^n}$.

Hence $(d(F_n)) \to 0$ as $n \to \infty$.

Since M is complete, by Cantor's intersection theorem, there exists a point$x$ in M such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Also, each F_n is disjoint from A_n.

Hence $x \notin A_n$ for all n.

$\therefore$ $x \notin \bigcup_{n=1}^{\infty} A_n$.

$\therefore$ $\bigcup_{n=1}^{\infty} A_n \neq M$. Hence M is of second category.

Corollary. $\mathbb{R}$ is of second category.

Proof. We know that $\mathbb{R}$ is a complete metric space. Hence $\mathbb{R}$ is of second category.

Note. The converse of the above theorem is not true.

(i.e.) A metric space which is of second category need not be complete.

For example, consider $M = \mathbb{R} - \mathbb{Q}$, the space of irrational numbers.

We know that $\mathbb{Q}$ is of first category.

Suppose M is of first category. Then $M \cup \mathbb{Q} = \mathbb{R}$ is also of first category which is contradiction.

Also M is not a closed subspace of $\mathbb{R}$ and hence M is not complete.

SOLVED PROBLEMS

Problem 1. Prove that any nonempty open interval (a, b) in $\mathbb{R}$ is of second category.

Solution. Let (a, b) be a non-empty open interval in $\mathbb{R}$.

Suppose (a, b) is of first category.

Now, $[a, b] = (a, b) \cup \{a\} \cup \{b\}$.

$\therefore$ $[a, b]$ is of first category.

But $[a, b]$ is a complete metric space and hence is of second category which is a contradiction.

$\therefore$ (a, b) is of second category.

Problem 2. Prove that a closed set A in a metric space M is nowhere dense iff A^c is everywhere dense.

Solution. Let A be a closed set in M.

$\therefore A = \bar{A}$. ....(1)

Suppose A is nowhere dense in M.

$\therefore Int\ \bar{A} = \Phi$.

$\therefore Int\ A = \Phi$. (by (1)) ......(2)

Now we claim that $\overline{A^c} = M$.

Obviously, $\overline{A^c} \subseteq M$. ....(3)

Now, let $x \in M$. Let G be any open set such that $x \in G$.

Since $Int\ A = \Phi$, we have $G \not\subset A$.

$\therefore G \cap A^c \neq \Phi$.

$\therefore x \in \overline{A^c}$.

$\therefore$ A^c is everywhere dense in M.

NOTES

Conversely let A^c be everywhere dense in M.

$\overline{A^c} = M$.

We claim that $Int\ A = \Phi$.

Let G be any non-empty open set in M.

Since $\bar{A} = M$, we have $G \cap A^c \neq \Phi$.

$\therefore G \not\subset A$.

$\therefore$ The only open set which is contained in A is the empty set.

$\therefore Int\ A = \Phi$.

$\therefore Int\ \bar{A} = \Phi$. (by (1))

$\therefore A$ is nowhere dense in M.

CHECK YOUR PROGRESS

1. If A and B are closed subset of $\mathbb{R}$ prove that $A \times B$ is a closed subset in $\mathbb{R} \times \mathbb{R}$.
2. Consider $\mathbb{R}$ with usual metric. Show that in any singleton set $\{x\}$ is nowhere dense.

5.4 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Since A and B are closed sets we have $A = \bar{A}$ and $B = \bar{B}$.
Now, $\overline{A \times B} = \bar{A} \times \bar{B} = A \times B$ (by problem 1). Therefore, $A \times B$ is a closed set.
2. Consider $\mathbb{R}$ with usual metric. Any singleton set $\{x\}$ is nowhere dense. Therefore, any countable subset of $\mathbb{R}$ begin a countable union of singleton sets is of first category. In particular $\mathbb{Q}$ is of first category. (refer theorem 3)

5.5 SUMMARY

1. Let (M, d) be a metric space. Let $(x_n) = x_1, x_2, \dots, x_n, \dots$ be a sequence of point in M. Let $x \in M$. We say (x_n) is **converges** to x if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ such that for all $n \geq n_0$. Also x is called a **limit** of (x_n).
2. $(x_n) \to x$ iff for each open ball $B(x, \varepsilon)$ with center x there exists a positive integer n_0 such that $x_n \in B(x, \varepsilon)$ for all $n \geq n_0$.
3. $(x_n) \to x$ iff the sequence of real numbers $(d(x_n, x)) \to 0$.
4. For a convergence sequence (x_n) the limit is unique.

NOTES

5. Let (M,d) be a metric space. Then any convergence sequence in M is a Cauchy sequence.
6. A subset A of a complete metric space M is complete iff A is closed.
7. Any complete metric space is of second category.

5.6 KEYWORDS

6. **Converges:** Let (M,d) be a metric space. Let $(x_n) = x_1, x_2, \dots, x_n, \dots$ be a sequence of point in M. Let $x \in M$. We say (x_n) is converges to x if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ such that for all $n \geq n_0$.
7. **Cauchy sequence**: Let (M,d) be a metric space. Let (x_n) be a sequence of points in M. (x_n) is said to be a Cauchy sequence in M if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_{0.}$
8. **Complete:** A metric space M is said to be complete if every Cauchy sequence in M converges to a point in M.
9. **Nowhere dense:** A subset A of a metric space M is said to be nowhere dense in M if $Int\ \bar{A} = \Phi$.
10. **First category:** A subset A of a metric space M is said to be of first category in M if A can be expressed as a countable union of nowhere dense sets.
11. **Second category:** A set which is not of first category is of second category.

5.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Show that $\mathbb{R}$ with usual metric is complete.
2. Show that [0,1] with usual metric is complete.
3. Prove that any discrete metric space is complete.
4. Show that $\mathbb{R}$ is of second category.
5. Prove that union of a countable number of sets which are of first category is again of first category.
6. Prove that $\mathbb{R}^n$ with each of the following metric is complete.

a. $d_1(x,y) = \max\{|x_i - y_i| \mid i = 1,2, \dots n\}$.

b. $d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i|$.

7. Prove that l_p is a complete metric space for any $p \geq 1$.

NOTES

5.8 FURTHER READINGS

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NOTES

UNIT-VI CONTINUITY

STRUCTURE

6.0 INTRODUCTION

In unit 5, we discussed the concept of convergence of a sequence in any metric space. The definition of continuity for real valued functions depends on the usual metric of the real line. Hence the concept of continuity can be extended for functions defined from one metric space to another in a natural way.

6.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by continuous.
- Determine homeomorphism.
- Discuss uniform continuity.

6.2 CONTINUITY

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces. Let $f: M_1 \to M_2$ be a function. Let $a_1 \in M_1$ and $l \in M_2$. The function f is said to have **limit** as $x \to a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < d_1(x, a) < \delta \Longrightarrow d_2(f(x), l) < \varepsilon$. We write $\lim_{x\to a} f(x) = l$.

NOTES

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is said to be **continuous** at a if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$.

f is said to be **continuous** if it is continuous at every point of M_1.

Note 1. f is continuous at a iff $\lim_{x \to a} f(x) = f(a)$.

Note 2. The continuous $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$ can be rewritten as

(i) $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$ or
(ii) $f\big(B(a, \delta)\big) \subseteq B(f(a), \varepsilon)$.

Example1. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then any constant function $f: M_1 \to M_2$ is continuous.

Proof. Let $f: M_1 \to M_2$ be given by $f(x) = a$, where $a \in M_2$ is a fixed element.

Let $x \in M_1$ and $\varepsilon > 0$ be given.

Then for any $\delta > 0, f\big(B(x, \delta)\big) = \{a\} \subseteq B(a, \varepsilon)$.

$\therefore f$ is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous.

Example 2. Let (M_1, d_1) be a discrete metric space and let (M_2, d_2) be any metric space. Then any function $f: M_1 \to M_2$ is continuous.

i.e. Any function whose domain is a discrete metric space is continuous.

Proof. Let $x \in M_1$ and $\varepsilon > 0$ be given.

Since M_1 is discrete for any $\delta < 1, B(x, \delta) = \{x\}$.

$$\therefore f\big(B(x, \delta)\big) = \{f(x)\} \subseteq B(f(x), \varepsilon)$$

$\therefore f$ is continuous at x.

We now give a characterization for continuity of a function at a point in terms of sequences converging to that point.

Theorem 1. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is continuous at a iff $(x_n) \to a \Rightarrow \big(f(x_n)\big) \to f(a)$.

NOTES

Proof. Suppose f is continuous at a. Let (x_n) be a sequence in M_1 such that $(x_n) \to a$.

We claim that $\left(f(x_n)\right) \to f(a)$.

Let $\varepsilon > 0$ be given. By the definition of continuity, there exists $\delta > 0$such that $d_1(x, a) < \delta \Longrightarrow d_2(f(x), f(a)) < \varepsilon$. .....(1)

Since $(x_n) \to a$, there exists a positive integer n_0 such that $d_1(x_n, a) < \delta$ for all $n \geq n_0$.

$\therefore d_2(f(x_n), f(a)) < \varepsilon$ for all $n \geq n_0$ (by (1))

$$\therefore \left(f(x_n)\right) \to f(a).$$

Conversely, suppose $(x_n) \to a \Longrightarrow \left(f(x_n)\right) \to f(a)$.

We claim that f is continuous at a.

Suppose f is not continuous at a.

Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$, $f\left(B\left(a, \frac{1}{n}\right)\right) \not\subset B(f(a), \varepsilon)$.

In particular $f\left(B\left(a, \frac{1}{n}\right)\right) \not\subset B(f(a), \varepsilon)$

Choose x_n such that $x_n \in B\left(a, \frac{1}{n}\right)$ and $f(x_n) \notin B(f(a), \varepsilon)$.

$\therefore d_1(x_n, a) < \frac{1}{n}$, and $d_2\left(f(x_n), f(a)\right) \geq \varepsilon$.

$\therefore (x_n) \to a$ and $(f(x_n))$ does not converges to $f(a)$ which is a contradiction to the hypothesis.

$\therefore f$ is continuous at a.

Corollary. A function $f: M_1 \to M_2$ is continuous iff $(x_n) \to x \Longrightarrow \left(f(x_n)\right) \to f(x)$.

We now characterize continuous mapping in terms of open sets.

Theorem 2. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. $f: M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

(i.e.) f is continuous iff inverse image of every open set is open.

Proof. Suppose f is continuous.

NOTES

Let G be an open set in M_2.

We claim that $f^{-1}(G)$ is open in M_1.

If $f^{-1}(G)$ is an empty, then it is open.

Let $f^{-1}(G) \neq \Phi$.

Let $x \in f^{-1}(G)$. Hence $f(x) \in G$.

Since G is open, there exists an open ball $B(f(x), \varepsilon)$ such that $B(f(x), \varepsilon) \subseteq G$. .....(1)

Now, by definition of continuity, there exists an open ball $B(x, \delta)$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$.

$\therefore f\big(B(x, \delta)\big) \subseteq G$. (By (1)).

$\therefore B(x, \delta) \subseteq f^{-1}(G)$.

Since $x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open.

Conversely, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2. We claim that f is continuous.

Let $x \in M_1$.

Now, $B(f(x), \varepsilon)$ is an open set in M_2.

$\therefore f^{-1}(B(f(x), \varepsilon))$ is open in M_1 and $x \in f^{-1}(B(f(x), \varepsilon))$.

$\therefore$ There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}\big(B(f(x), \varepsilon)\big)$.

$$\therefore f(B(x, \delta)) \subseteq B(f(x), \varepsilon).$$

$\therefore f$ is continuous at x.

Since $x \in M_1$ is arbitrary f is continuous.

Note 1. If $f: M_1 \to M_2$ is continuous and G is open in M_1, then it is not necessary that $f(G)$ is open in M_2

(i.e.) Under a continuous map the image of an open set need not be an open set.

For example let $M_1 = \mathbb{R}$ with discrete metric and let $M_2 = \mathbb{R}$ with usual metric.

Let $f: M_1 \to M_2$ be defined by $f(x) = x$.

Since M_1 is discrete every subset of M_1 is open.

Hence for any open subset G of M_2, $f^{-1}(G)$ is open in M_1.

$\therefore f$ is continuous.

Now, $A = \{x\}$ is open in M_1.

But $f(A) = \{x\}$ is not open in M_2.

NOTES

Note 2. In the above example f is a continuous bijection whereas $f^{-1}: M_1 \to M_2$ is not continuous.

For, $\{x\}$ is an open set in M_1.

$(f^{-1})^{-1}(\{x\}) = \{x\}$ which is not open in M_2.

$\therefore f^{-1}$ is not continuous.

Thus if f is a continuous bijection, f^{-1} need not be continuous.

We now give yet another characterization of continuous functions in terms of closed sets.

Theorem 3. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f: M_1 \to M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is open in M_2.

Proof. Suppose $f: M_1 \to M_2$ is continuous.

Let $F \subseteq M_2$ be an closed in M_2.

$\therefore F^c$ is open in M_2.

$\therefore f^{-1}(F^c)$ is open in M_1.

But $f^{-1}(F^c) = [f^{-1}(F)\,]^c$.

$f^{-1}(F)$ is closed in M_1.

Conversely, suppose $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2. We claim that f is continuous.

Let G is an open set in M_2.

$\therefore G^c$ is closed in M_2.

$\therefore f^{-1}(G^c)$ is closed in M_1.

$\therefore [f^{-1}(G^c)]^c$ is closed in M_1.

$\therefore f^{-1}(G)$ is open in M_1.

NOTES

$\therefore f$ is continuous.

We give one more characterization of continuous function in terms of closure of a set.

Theorem 4. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f: M_1 \to M_2$ is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Proof. Suppose f is continuous.

Let $A \subseteq M_1$. Then $f(A) \subseteq M_2$.

Since f is continuous, $f^{-1}(\overline{f(A)}\,)$ is closed in M_1.

Also, $f^{-1}(\overline{f(A)}\,) \supseteq A$ (since $\overline{f(A)} \supseteq f(A)$).

But $\bar{A}$ is the smallest closed set containing A.

$\therefore f^{-1}(\overline{f(A)}\,) \supseteq \bar{A}$.

$\therefore f(\bar{A}) \subseteq \overline{f(A)}$.

Conversely, let $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove that f continuous, we shall show that if F is a closed set in M_2, then $f^{-1}(F)$ is closed in M_1.

By hypothesis, $f(\overline{f^{-1}(F)}) \subseteq \overline{ff^{-1}(F)}$

$\subseteq \bar{F}$.

$= F$ (Since F is closed).

Thus $f\left(\overline{f^{-1}(F)}\right) \subseteq F$.

$\therefore \overline{f^{-1}(F)} \subseteq F$.

Also
$\overline{f^{-1}(F)} \subseteq \overline{f^{-1}(F)}$.

$\therefore f^{-1}(F) = \overline{f^{-1}(F)}$.

Hence $f^{-1}(F)$ is closed.

$\therefore f$ is continuous.

NOTES

Solved problems

Problem1. Let f be a continuous real valued function defined on a metric space M. Let $A = \{x \in M | f(x) \geq 0\}$. Prove that A is closed.

Solution. $A = \{x \in M | f(x) \geq 0\}$.

$$= \{x \in M | f(x) \in [0, \infty)\}.$$

$$= f^{-1}([0, \infty)).$$

Also, $[0, \infty)$ is a closed subset of $\mathbb{R}$.

Since f is continuous, $f^{-1}([0, \infty))$ is closed in M.

$\therefore$ A is closed.

Problem 2. Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 \; if \; x \; is \; irrational \\ 1 \; if \; x \; is \; rational \end{cases}$$

is not continuous by each of the following methods.

(i) By the usual ε, δ method.

(ii) By exhibiting a sequence (x_n) such that $(x_n) \to x$ and $(f(x_n))$ does not converge to $f(x)$.

(iii) By exhibiting an open set G such that $f^{-1}(G)$ is not open.

(iv) By exhibiting closed subset F such that $f^{-1}(F)$ is not closed.

(v) By exhibiting an subset A of $\mathbb{R}$ such that $f(\bar{A}) \not\subset \overline{f(A)}$.

Solution (i). To prove that f is not continuous at x we have to show that there exists an $\varepsilon > 0$ such that for all $\delta > 0$, $f(B(x, \delta)) \not\subset B(f(x), \varepsilon)$.

Let $\varepsilon = \frac{1}{2}$.

For any $\delta > 0, B(x, \delta) = (x - \delta, x + \delta)$ contains both rational and irrational numbers.

If x is rational, choose $y \in B(x, \delta)$ such that y is rational.

Then $|f(x) - f(y)| = 1$. (by definition of f).

(i.e.) $d(f(x), f(y)) = 1$.

$\therefore f(y) \notin B(f(x), \frac{1}{2})$

NOTES

Thus $y \in B(x,\delta)$ and $f(y) \notin B(f(x),\frac{1}{2})$

$\therefore f\big(B(x,\delta)\big) \not\subset B(f(x),\varepsilon)$.

Hence f is not continuous at x.

(ii) Let $x \in \mathbb{R}$. Suppose x is rational. Then $f(x) = 1$. Let (x_n) be a sequence of irrational numbers such that $(x_n) \to x$.

Then $(f(x_n)) \to 0$ and $f(x) = 1$.

$\therefore (f(x_n))$ does not converge to $f(x)$.

Proof is similar if x is irrational.

(iii) Let $G = \left(\frac{1}{2},\frac{3}{2}\right)$. Clearly G is open in $\mathbb{R}$.
Now, $f^{-1}(G) = \{x \in \mathbb{R} | f(x) \in G\}$.

$$= \left\{x \in \mathbb{R} \middle| f(x) \in \left(\frac{1}{2},\frac{3}{2}\right)\right\}.$$

$$= \mathbb{Q}.$$

But $\mathbb{Q}$ is not open in $\mathbb{R}$.
Thus $f^{-1}(G)$ is not open in $\mathbb{R}$.
$\therefore f$ is continuous.

(iv) Choose $F = [\frac{1}{2},\frac{3}{2}]$
Then, $f^{-1}(F) = \mathbb{Q}$ which is not closed in $\mathbb{R}$.
$\therefore f$ is not continuous.

(v) Let $A = \mathbb{Q}$. Then $\bar{A} = \mathbb{R}$. (refer example1)
$\therefore f(\bar{A}) = f(\mathbb{R}) = \{0,1\}$ (by definition of f).
Also, $f(A) = f(\mathbb{Q}) = \{1\}$.
$\therefore \overline{f(A)} = \overline{\{1\}} = \{1\}$.
$\therefore \overline{f(A)} \not\subset f(\bar{A})$.
$\therefore f$ is not continuous.

Problem 3. Let M_1, M_2, M_3 be metric spaces. If $f: M_1 \to M_2$ and $g: M_2 \to M_3$ are continuous functions, prove that $g \circ f: M_1 \to M_3$ is also continuous.

(i.e.) composition of two continuous functions is continuous.

Solution. Let G be open in M_3.

Since g is continuous, $g^{-1}(G)$ is open in M_2.

Now, since f is continuous, $f^{-1}(g^{-1}(G)\ )$ is open in M_1.

(i.e.) $(g \circ f)^{-1}(G)$ is open in M_1.

NOTES

$\therefore g \circ f$ is continuous.

Problem 4. Let M be a metric space. Let $f: M \to \mathbb{R}$ and $g: M \to \mathbb{R}$ be two continuous functions. Prove that $f + g: M \to \mathbb{R}$ is continuous.

Solution. Let (x_n) be a sequence converging to x in M.

Since f and g are continuous functions, $(f(x_n)) \to f(x)$ and , $\big(g(x_n)\big) \to g(x)$.

$\therefore \big(f(x_n) + g(x_n)\big) \to f(x) + g(x)$.

(i.e.) , $\big((f+g)(x_n)\big) \to (f+g)(x)$

$\therefore f + g$ is continuous.

Problem 5. Let f, g be continuous real valued functions on a metric space M. Let $A = \{x | x \in M \; and \; f(x) < g(x)\}$. Prove that A is open.

Solution. Since f and g are continuous real valued function on $M, f - g$ is also a continuous real valued function on M.

Now $A = \{x \in M | \; f(x) < g(x)\}$.

$= \{x \in M | \; f(x) - g(x) < 0\}$.

$= \{x \in M | \; (f - g)(x) < 0\}$

$= \{x \in M | \; (f-g)x \in (-\infty, 0)\}$
.
$= (f-g)^{-1}\{(-\infty, 0)\}$.
Now, $(-\infty, 0)$ is open in $\mathbb{R}$, and $f - g$ is continuous.
Hence $(f-g)^{-1}\{(-\infty, 0)\}$ is open in M.
$\therefore A$ is open in M.

Problem 6. If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ and if $h: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $h(x, y) = (f(x), g(y))$ prove that h is continuous on $\mathbb{R}^2$.

Solution. Let (x_n, y_n) be sequence in $\mathbb{R}^2$ converging to (x, y).

We claim that $(h(x_n, y_n))$ converges to $h(x, y)$.

Since $\big((x_n, y_n)\big) \to (x, y)$ in $\mathbb{R}^2$, $(x_n) \to x$ and $(y_n) \to y$ in $\mathbb{R}$.

Also f and g are continuous.

$\therefore (f(x_n)) \to f(x)$ and $\big(g(y_n)\big) \to g(y)$.

NOTES

$\therefore \big(f(x_n), g(y_n)\big) \to (f(x), g(y))$.

$$\therefore \big(h(x_n, y_n)\big) \to h(x, y).$$

$\therefore h$ is continuous on $\mathbb{R}^2$.

Problem 7. Let (M, d) be a metric space. Let $a \in M$. Show that the function $f: M \to \mathbb{R}$ defined by $f(x) = d(x, a)$ is continuous.

Solution. Let $x \in M$.

Let (x_n) be a sequence in M such that $(x_n) \to x$.

We claim that $\big(f(x_n)\big) \to f(x)$.

Let $\varepsilon > 0$ be given.

Now, $|f(x_n) - f(x)| = |d(x_n, a) - d(x, a)| \leq d(x_n, x)$.

Since, $(x_n) \to x$, then there exists a positive integer n_1 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_1$.

$\therefore |f(x_n) - f(x)| < \varepsilon$ for all $n \geq n_1$.

$\therefore \big(f(x_n)\big) \to f(x)$.

$\therefore f$ is continuous.

Problem 8. Let f be a function from $\mathbb{R}^2$ onto $\mathbb{R}$ defined by $f(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Show that f is continuous in $\mathbb{R}^2$.

Solution. Let $(x, y) \in \mathbb{R}^2$.

Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$converging to (x, y).

Then $(x_n) \to x$ and $(y_n) \to y$.

$\therefore \big(f(x_n, y_n)\big) = (x_n) \to x = f(x, y)$.

$\therefore \big(f(x_n, y_n)\big) \to f(x, y)$.

$\therefore f$ is continuous.

Problem 9. Define $f: l_2 \to l_2$ as follows. If $s \in l_2$ is the sequence $s_1, s_2, \ldots.$ let $f(s)$ be the sequence $0, s_1, s_2, \ldots.$ Show that f is continuous on l_2.

Solution. Let $y = (y_1, y_2, \ldots., y_n, \ldots) \in l_2$.

Let (x_n) be a sequence in l_2 converging to y.

NOTES

Let $x_n = \left(x_{n_1}, x_{n_2}, \ldots., x_{n_k}, \ldots\right)$.

Then $(x_{n_1}) \to y_1$, $(x_{n_2}) \to y_2$, ….,$(x_{n_k}) \to y_k$,…..

$\therefore \left(f(x_n)\right) = \left(\left(0, x_{n_1}, x_{n_2}, \ldots., x_{n_k}, \ldots\right)\right) \to (0, y_1, y_2, \ldots., y_k, \ldots)) = f(y)$.

$\therefore \left(f(x_n)\right) \to f(y)$.

$\therefore f$ is continuous.

Problem 10. Let G be an open subset of $\mathbb{R}$. Prove that the characteristic function on G defined by $\chi_G(x) = \begin{cases} 1 \; if \; x \in G \\ 0 \; if \; x \notin G \end{cases}$ is continuous at every point of G.

Solution. Let $x \in G$ so that $\chi_G(x) = 1$.

Let $\varepsilon > 0$ be given.

Since G is open and $x \in G$, we can find a $\delta > 0$ such that $B(x, \delta) \subseteq G$.

$\therefore \chi_G\left(B(x, \delta)\right) \subseteq \chi_G(G)$.

$= \{1\}$.

$\subseteq B(1, \varepsilon)$.

Thus $\chi_G\left(B(x, \delta)\right) \subseteq B(\chi_G(x), \varepsilon)$.

$\therefore \chi_G$ is continuous at x.

Since $x \in G$ is arbitrary, χ_G is continuous on G.

6.3 HOMEOMORPHISM

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f: M_1 \to M_2$ is called a **homeomorphism** if

(i) f is 1-1 and onto.
(ii) f is continuous.
(iii) f^{-1} is continuous.

M_1 and M_2 are said to be **homeomorphic** if there exists a homeomorphism $f: M_1 \to M_2$.

Definition. A function $f: M_1 \to M_2$ is said to be an **open map** if $f(G)$ is open in M_2 for every open set G in M_1.

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(i.e.) f is an open map if the image of an open set in M_1 is an open set in M_2.

f is called a **closed map** if $f(F)$ is closed in M_2 for every closed set F in M_1.

Note 1. Let $f: M_1 \to M_2$ be a 1-1 onto function. Then f^{-1} is continuous iff f is an open map.

For, f^{-1} is continuous iff for any open set G in $M_1 (f^{-1})^{-1}(G)$ is open in M_2.

But $(f^{-1})^{-1}(G) = f(G)$.

$\therefore f^{-1}$ is continuous iff for every open set G in M_1, $f(G)$ is open in M_2.

$\therefore f^{-1}$ is continuous iff f is an open map.

Note 2. Similarly f^{-1} is continuous iff f is a closed map.

Note 3. Let $f: M_1 \to M_2$ be a 1-1 onto map. Then the following are equivalent.

(i) f is homeomorphism.
(ii) f is continuous open map.
(iii) f is a continuous closed map.

Proof. **(i) ⇔(ii)** follows form Note 1 and the definition of homeomorphism.

(i) ⇔(iii) follows form Note 2 and the definition of homeomorphism.

Note 4. Let $f: M_1 \to M_2$ be a homeomorphism. $G \subseteq M_1$ is open in M_1 iff $f(G)$ is open in M_2.

For, since f is an open map G is open in $M_1 \Rightarrow f(G)$ is open in M_2.

Also since f is continuous, $f(G)$ is open in $M_2 \Rightarrow f^{-1}\big(f(G)\big) = G$ is open in M_1.

$\therefore G$ is open in M_1 iff $f(G)$ is open in M_2 . ... (1)

Conversely, if $f: M_1 \to M_2$ is a 1-1 onto map satisfying (1) then f is homeomorphism.

Thus a homeomorphism $f: M_1 \to M_2$ is simply a 1-1 onto map between the points of the two spaces such that their open sets are also in 1-1 correspondence with each other.

Note 5. Let $f: M_1 \to M_2$ be a 1-1 onto map. Then f is a homeomorphism iff it satisfies the following condition.

F is closed in M_1 iff $f(F)$ is closed in M_2 .

Example 1. The metric spaces [0,1] and [0,2] with usual metric are homeomorphic.

Proof. Define $f: [0,1] \to [0,2]$ by $f(x) = 2x$.

Clearly, f is $1 - 1$ and onto.

Also $f^{-1}(x) = \frac{1}{2}x$.

We note that f and f^{-1} are both continuous.

$\therefore$ f is homeomorphism.

Example 2. The metric spaces $(0, \infty)$ and $\mathbb{R}$ with usual metrics are homeomorphic.

Proof. $f: (0, \infty) \to \mathbb{R}$ by $f(x) = \log_e x$ is the required homeomorphism. Here $f^{-1}(x) = e^x$.

Example 3. The metric spaces $(\frac{-\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$ with usual metric are homeomorphic and $f: (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(x) = \tan x$ is the required homeomorpism.

In this example, $(\frac{-\pi}{2}, \frac{\pi}{2})$ is not a complete metric space whereas $\mathbb{R}$ is complete.

This shows that *completeness of metric spaces is not preserved under homeomorphism.*

Example 4. The metric spaces (0,1) and $(0, \infty)$ with usual metrics are homeomorphism.

Proof. Define $f: (0,1) \to (0, \infty)$ by $f(x) = \frac{x}{1-x}$.

We claim that f is 1-1 and onto.

Let $f(x) = f(y)$.

$\therefore \frac{x}{1-x} = \frac{y}{1-y}$.

$\therefore x - xy = y - xy$.

$\therefore x = y$. Hence f is 1-1.

NOTES

Let $y \in (0, \infty)$.

$\therefore f(x) = y \Rightarrow \frac{x}{1-x} = y$.

$\Rightarrow y - xy = x$.

$\Rightarrow x(1 + y) = y$.

$\Rightarrow x = \frac{y}{1+y}$.

$\therefore \frac{y}{1+y} \in (0,1)$ is the preimage of y under f.

Clearly f and f^{-1} are continuous.

$\therefore f$ is homeomorphism.

Example 5. $\mathbb{R}$ with usual metric is not homeomorphic to $\mathbb{R}$ with discrete metric.

Proof. Let $M_1 = \mathbb{R}$ with usual metric.

Let $M_2 = \mathbb{R}$ with discrete metric.

Let $f: M_1 \to M_2$ be any 1-1 onto map.

Now, $\{a\}$ is open in M_2.

But $f^{-1}(\{a\}) = \{ f^{-1}(a)\}$ is not open in M_1.

Hence f is not continuous.

Thus any bijection $f: M_1 \to M_2$ is not a homeomorphism.

Hence M_1 is not homeomorphism.

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

Let $f: M_1 \to M_2$ be a 1-1 onto map. f is said to be an **isometry** if $d_1(x, y) = d_2\big(f(x), f(y)\big)$ for all $x, y \in M_1$. In other words, an *isometry is a distance preserving map.*

M_1 and M_2 are said be **isometric** if there exists an isometry f from M_1 onto M_2.

Example 6. $\mathbb{R}^2$ with usual metric and $\mathbb{C}$ with usual metric are isometric and $f: \mathbb{R}^2 \to \mathbb{C}$ defined by $f(x, y) = x + iy$ is the required isometry.

Proof. Let d_1 denote the usual metric on $\mathbb{R}^2$ and d_2 denote the usual metric on $\mathbb{C}$.

NOTES

Let $a = (x_1, y_1)$ and $b = (x_2, y_2) \in \mathbb{R}^2$.

Then $d_1(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$= |(x_1 - x_2) + i(y_1 - y_2)|$$

$$= |(x_1 + iy_1) + (x_2 + i\,y_2)|$$

$$= d_2(f(a), f(b))$$

$\therefore f$ is an isometry.

Note. Since an isometry f preserves distances, the image of an open ball $B(x, r)$ is the open ball $B(f(x), r)$.

Hence it follows that under an isometry the image of an open set is also an open set. Also if f is an isometry f^{-1} is also an isometry.

Hence under an isometry the inverse image of an open set is open. Hence an isometry is a homeomorphism.

However a homeomorphism from one metric space to another need not be an isometry.

For example, $f: [0,1] \to [0,2]$ defined by $f(x) = 2x$ is a homeomorphism. (refer example 1)

But f is not an isometry. (refer example 6).

6.4 UNIFORM CONTINUITY

Introduction. In this section we introduce the concept of uniform continuity.

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $f: M_1 \to M_2$ be a continuous function. For each $a \in M_1$ the following is true. Given $\varepsilon > 0$, there exist $\delta > 0$ such that

$$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon.$$

In general the number δ depends on ε and the point a under considertation.

For example, consider $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

Let $a \in \mathbb{R}$. Let $\varepsilon > 0$ be given.

We want to find $\delta > 0$ such that

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$ ...... (1)

NOTES

Clearly, if $\delta > 0$ satisfies (1), then any δ_1 where $0 < \delta_1 < \delta$ also satisfies (1).

Hence if there exists a $\delta > 0$ satisfying (1), then we can find another δ_1 such that $0 < \delta_1 < 1$ and δ_1 also satisfies (1).

Hence we may restrict x such that $|x - a| < 1$.

$$\therefore a - 1 < x < a + 1.$$

$$\therefore x + a < 2a + 1.$$

$$\therefore |f(x) - f(a)| = |x^2 - a^2| = |x + a||x - a|$$

$< |2a + 1||x - a|$ if $|x - a| < 1$.

Hence if we choose $\delta = \min\{1, \frac{\varepsilon}{|2a+1|}\}$ then we have $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Thus, in this example we see that the number δ depends on both ε and the point a under consideration and if a become larger, δ has to be chosen correspondingly small. In fact, there is no $\delta > 0$ such that (1) holds for all a.

For, suppose there exists $\delta > 0$ such that

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$ for all $a \in \mathbb{R}$.

Take $x = a + \frac{1}{2}\delta$.

Clearly, $|x - a| = \frac{1}{2}\delta < \delta$.

$\therefore |f(x) - f(a)| < \varepsilon$.

$$\therefore \left|\left(a + \frac{1}{2}\delta\right)^2 - a^2\right| < \varepsilon.$$

$$\therefore \frac{1}{2}\delta\left|\frac{1}{2}\delta + 2a\right| < \varepsilon.$$

However this equality cannot be true for all $a \in \mathbb{R}$, since by taking a sufficiently large, we can make $\frac{1}{2}\delta\left|\frac{1}{2}\delta + 2a\right| > \varepsilon$.

Thus, there is no $\delta > 0$ such that (1) holds for all $a \in \mathbb{R}$.

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$.

Let $a \in \mathbb{R}$. Let $\varepsilon > 0$ be given.

NOTES

Then $|f(x) - f(a)| = |2x - 2a| = 2|x - a|$.

$\therefore$ If we choose $\delta = \frac{1}{2}\varepsilon$ then we have $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Here δ depends on ε and not an a.

(i.e.) for a given $\varepsilon > 0$ we are able to find $\delta > 0$ such that δ works *uniformly* for all $a \in \mathbb{R}$.

Definition. Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

A function $f: M_1 \to M_2$ be a **uniformly continuous** on M_1 if given $\varepsilon > 0$, there exist $\delta > 0$ such that

$$d_1(x, y) < \delta \Rightarrow d_2\big(f(x), f(y)\big) < \varepsilon.$$

Note 1. Uniformly continuity is a global condition on the behavior of a mapping on a set so that it is meaningless to ask whether a function is uniformly continuous at a point. Continuity is a local condition of the behavior of a function at a point.

Note 2. If $f: M_1 \to M_2$ is uniformly continuous on M_1 then f is continuous at every point of M_1.

Moreover for a given $\varepsilon > 0$, there exist $\delta > 0$ such that $x, y \in M_1$ and

$$d_1(x, y) < \delta \Rightarrow d_2\big(f(x), f(y)\big) < \varepsilon.$$

Thus, uniformly continuity is a continuity plus the added condition that for a given $\varepsilon > 0$ we can find $\delta > 0$ which works *uniformly for all points of* M_1.

$$d_1(x, y) < \delta \Rightarrow d_2\big(f(x), f(y)\big) < \varepsilon.$$

Note 3. A continuous function $f: M_1 \to M_2$ need not be uniformly continuous on M_1.

For example, $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous but not uniformly continuous $\mathbb{R}$.

Solved problems

Problem 1. Prove that $f: [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous on $[0,1]$.

Solution. Let $\varepsilon > 0$ be given. Let $x, y \in [0,1]$.

Then

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y|$$

NOTES

$\leq |x-y|$ (since $x \leq 1$ and $y \leq 1$)

$$\therefore |x-y| < \frac{1}{2}\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon.$$

$\therefore f$ is uniformly continuous on $[0,1]$.

Problem 2. Prove that the function $f: [0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Solution. Let $\varepsilon > 0$ be given. Suppose there exist $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Take $x = y + \frac{1}{2}\delta$.

Clearly $|x-y| = \frac{1}{2}\delta < \delta$.

$\therefore |f(x) - f(a)| < \varepsilon$.

$\therefore \frac{1}{x} - \frac{1}{y} < \varepsilon$.

$\therefore \left|\frac{1}{y+\frac{1}{2}\delta} - \frac{1}{y}\right| < \varepsilon$.

$\therefore \left|\frac{1}{2(y+\frac{1}{2}\delta)} - \frac{1}{y}\right| < \varepsilon$.

$\therefore \left|\frac{\delta}{(2y+\delta)y}\right| < \varepsilon$.

This inequality cannot be true for all $y \in (0,1)$ since $\frac{\delta}{(2y+\delta)y}$ becomes arbitrarily large as y approaches zero.

$\therefore f$ is not uniformly continuous.

Problem 3. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is uniformly continuous $\mathbb{R}$.

Solution. Let $x, y \in \mathbb{R}$ and $x > y$.

$\sin x - \sin y = (x-y)\cos z$ where $x > z > y$ (by mean value theorem)

$$\therefore |\sin x - \sin y| = |x-y||\cos z|$$

$\leq |x-y|$ (since $|cos\ z| \leq 1$).

Hence for a given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, we have

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| = |\sin x - \sin y| < \varepsilon.$$

NOTES

$\therefore f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

CHECK YOUR PROGRESS

1. Let d_1 be the usual metric on $[0,1]$ and d_2 be the usual metric on $[0,2]$. The map $f: [0,1] \to [0,2]$ defined by $f(x) = 2x$ is not an isometry.
2. Define open map.

6.5 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Let $x, y \in [0,1]$. Then $d_2\big(f(x), f(y)\big) = |f(x) - f(y)| = |2x - 2y|$

   $= 2|x - y| = 2d_1(x, y)$. Therefore, $d_1(x, y) \neq d_2(f(x), f(y))$.

   Hence f is not an isometry.

   2. A function $f: M_1 \to M_2$ is said to be an **open map** if $f(G)$ is open in M_2 for every open set G in M_1.

6.6 SUMMARY

7. Let (M, d) be a metric space. Let $(x_n) = x_1, x_2, \dots, x_n, \dots$ be a sequence of point in M. Let $x \in M$. We say (x_n) is **converges** to x if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ such that for all $n \geq n_0$. Also x is called a **limit** of (x_n).
8. f is said to be **continuous** if it is continuous at every point of M_1.
9. f is continuous at a iff $\lim_{x \to a} f(x) = f(a)$.
10. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then any constant function $f: M_1 \to M_2$ is continuous.
11. Let (M_1, d_1) be a discrete metric space and let (M_2, d_2) be any metric space. Then any function $f: M_1 \to M_2$ is continuous.
12. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is continuous at a iff $(x_n) \to a \Rightarrow \big(f(x_n)\big) \to f(a)$.
13. f is continuous iff inverse image of every open set is open.
14. f is an open map if the image of an open set in M_1 is an open set in M_2.

6.7 KEYWORDS

1. **Limit:** Let (M_1, d_1) and (M_2, d_2) be metric spaces. Let $f: M_1 \to M_2$ be a function. Let $a_1 \in M_1$ and $l \in M_2$. The function f is said to have limit as $x \to a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), l) < \varepsilon$. We write $\lim_{x \to a} f(x) = l$.

NOTES

2. **Continuous:** Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is said to be continuous at a if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(x, a) < \delta \Longrightarrow d_2(f(x), f(a)) < \varepsilon$.

3. **Homeomorphism:** Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f: M_1 \to M_2$ is called a homeomorphism if (i)f is 1-1 and onto. (ii) f is continuous. (iii) f^{-1} is continuous.

4. **Homeomorphic:** M_1 and M_2 are said to be homeomorphic if there exists a homeomorphism $f: M_1 \to M_2$.

5. **Open map:** A function $f: M_1 \to M_2$ is said to be an open map if $f(G)$ is open in M_2 for every open set G in M_1.

6. **Closed map:** If is called a closed map if $f(F)$ is closed in M_2 for every closed set F in M_1.

7. **Uniformly continuous:** Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

8. A function $f: M_1 \to M_2$ be a uniformly continuous on M_1 if given $\varepsilon > 0$, there exist $\delta > 0$ such that $d_1(x, y) < \delta \Rightarrow d_2\big(f(x), f(y)\big) < \varepsilon$.

6.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Show that any function whose domain is discrete metric space is continuous.

2. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = f(x, y)$ show that f is continuous on $\mathbb{R}^2$.

3. Prove that any two open intervals are homeomorphic.

6.9 FURTHER READINGS

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UNIT-VII DIFFERENTIABLE FUNCTION

STRUCTURE

NOTES

7.0 INTRODUCTION

In this chapter, we shall introduce the notion of the derivation of the function and the properties of such functions. We shall consider only real valued functions defined on intervals.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by differentiability of functions.
- Determine derivability and continuity.
- Discuss algebra of derivatives.

NOTES

7.2 DIFFERENTIABILITY OF A FUNCTION

Definition. Let f be a be a real valued function defined on an interval $I \subset \mathbb{R}$. If $a \in I$, then f is said to have a derivative at $x = a$, if $\lim_{x \to a} \frac{f(x)-f(a)}{h}$ exists. If this limit exists, then f is said to be differentiable at a and its derivative is denoted by $f'(a)$. Note that this limit is a real number. If we make the substitution $h = x - a$, then the above limit can also be written as $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Thus if E is the set of points of I at which $f'(a)$ exists and $E \neq \Phi$, then f' is itself a real valued function on E. If f' is defined on E. If f' is defined at every point of E, then f is said to be differentiable on E. It is possible that $E \neq \Phi$ and there are functions which are differentiable at some points in the domain but not at other points of the domain.

7.3 DERIVABILITY & CONTINUITY

Theorem 1. If the real valued function f is differentiable at the point $a \in \mathbb{R}$, then f is continuous at a.

Proof. We know that f is continuous at $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$ or equivalently $\lim\limits_{x \to a} [f(x) - f(a)] = 0$.

For, $x \neq a$, we have $f(x) - f(a) = \frac{f(x)-f(a)}{(x-a)}(x - a)$.

Since $\lim_{x \to a} \frac{f(x)-f(a)}{(x-a)} = f'(a)$ and $\lim\limits_{x \to a}(x - a) = 0$.

We get $\lim\limits_{x \to a}[f(x) - f(a)] = f'(a).0 = 0$.

Therefore, if f is differentiable at $x = a$, then it is continuous at $x = a$.

The converse of the above theorem is false. There exists functions continuous at a point but not differentiable at the point. We shall illustrate this by an example.

NOTES

Example 2. Let $f(x) = |x|$ for $x \in (-\infty, \infty)$. This function is continuous everywhere and in particular it is continuous at $x = 0$.

If $x > 0, f(x) - f(0) = x$ and if $x < 0, f(x) - f(0) = -x$.

Hence, we have $\frac{f(x)-f(0)}{x-0} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{f(x)-f(0)}{x-0} = \frac{-x}{x} = -1$ if $x < 0$.

Therefore, $lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist. Thus f does not have a derivative at 0, even though f is continuous at 0.

Example 3. Let $f(x) = x|x|$ for $x \in \mathbb{R}$ then $f'(x) = 2|x|$ for every x in $\mathbb{R}$.

From the definition of the function, $f(x) = x^2$ if $x > 0$ and $f(x) = -x^2$ if $x < 0$. If $a > 0$, then we have $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2-(a)^2}{h}$.

Since $a + h > 0$, when $|h|$ is sufficiently small,

$$f'(a) = \lim_{h\to 0} \frac{2ah-h^2}{h} = \lim_{h\to 0}(2a + h) = 2a.$$

If $a > 0$, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{-(a+h)^2-(a)^2}{h}$$

Since $a + h < 0$, when $|h|$ is sufficiently small,

$$f'(a) = \lim_{h\to 0} \frac{-2ah-h^2}{h} = \lim_{h\to 0}(-2a - h) = -2a.$$

Let us consider the case when $x = 0$,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0}|h| = 0.$$

Combining all the above three cases, we get $f'(x) = 2|x|$ for every x in $\mathbb{R}$.

NOTES

Note. The function f' may have a derivative denoted by f'' which is defined at all points where f' is differentiable. f'' is called the second derivative of f.

7.4 ALGEBRA OF DERIVATIVES

The next theorem gives the different formulae for differentiating the sum, difference, product and quotient of two functions.

Theorem 4. If f and g are both differentiable at $x = a$ in $\mathbb{R}$, then $f + g, f - g$and fg are differentiable and have derivatives given by

(i) $(f+g)'(a) = f'(a) + g'(a)$.

(ii) $(f-g)'(a) = f'(a) - g'(a)$.

(iii) $(fg)'(a) = f'(a)\, g(a) + f(a)\, g'(a)$.

(iv) Furthermore, if $g'(a) \neq 0$, then f/g is differentiable at a and has derivative given by

$$\left(\frac{f}{g}\right)' = \frac{g(a)f'(a) - f(a)\, g'(a)}{[g(a)]^2}.$$

Proof. We shall prove (iii) and (iv), since (i) and (ii) can be proved easily. To prove (iii), let $h = fg$. Then for $x \neq a$, we get

$$h(x) - h(a) = f(x)g(x) - f(a)g(a)$$

$$= f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)$$

And so, $\frac{h(x)-h(a)}{x-a} = \frac{f(x)-f(a)}{x-a} g(x) + f(a) \frac{g(x)-g(a)}{x-a}$

Since $\lim_{x\to a} \frac{f(x)-f(a)}{(x-a)} = f'(a)$, $\lim_{x\to a} \frac{g(x)-g(a)}{(x-a)} = g'(a)$

By Theorem 1, $\lim\limits_{x\to a} g(x) = g(a)$. Hence, by using the theorem on limits, h has a derivative at a and

$$h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{(x-a)} = f'(a)\, g(a) + f(a)\, g'(a).$$

To prove (iv), let $h = f/g$. Then we have,

$$\frac{h(x)-h(a)}{(x-a)} = \frac{1}{g(x)g(a)}[g(a)\frac{f(x)-f(a)}{(x-a)} - f(a)\frac{g(x)-g(a)}{(x-a)}].$$

Since $f(x)$ and $g(x)$ are differentiable at a having the derivatives $f'(a)$ and $g'(a)$ and when $\neq 0$, $\lim_{x\to a} g(x) = g(a)$, we get from the above

$$lim_{x\to a}\frac{h(x)-h(a)}{(x-a)} = \frac{g(a)f'(a)-f(a)g'(a)}{g^2(a)}.$$

Example 5. The derivative of any constant is zero. If $f(x) = x$, then $f'(x) = 1$. By using (iii) repeatedly we see that x^n is differentiable and the derivative is nx^{n-1} for any integer n, when $x \neq 0$. Thus a polynomial is differentiable and using (iv) repeatedly we see that every rational function is differentiable except at the point where the denominator is zero.

For two functions f and g, the composite function $h = f \circ g$ is defined at each point a, $h(a) = (g \circ f)(a) = g[f(a)]$.

Theorem 6. If f is differentiable at a, and g is differentiable at $f(a)$, then $h = f \circ g$ is differentiable at a and has the derivative $h'(a) = g'[f(a)]f'(a)$.

Proof. Let f be differentiable at a and g be differentiable at $b = f(a)$. It is assumed that f is defined in some neighbourhood of a and that g is defined in some neighbourhood of $b = f(a)$.

f is continuous at a and g is continuous at $b = f(a)$. Thus $h = f \circ g$ is continuous at $x = a$.

Let us define $\eta(x) = \frac{f(x)-f(a)}{x-a} - f'(a)$.

Since $\lim_{x\to a}\frac{f(x)-f(a)}{(x-a)}$ exists, $\eta(x)$ exists in a deleted neighbourhood of a and $\eta(x) \to 0$ as $x \to a$. Hence, if f is differentiable at a, we can write

$$f(x) - f(a) = (x-a)[f'(a) + \eta(x)].$$

NOTES

Similarly, since g is differentiable at $b = f(a)$. We have $g(y) - g(b) = (y - b)[g'(b) + \gamma(y)]$ where $\gamma(y) \to 0$ as $y \to b$.

Now, we have

$$h(x) - h(a) = (g \circ f)x - (g \circ f)a = g[f(x)] - g[f(a)].$$

$= g(y) - g(b) = (y - b)[g'(b) + \gamma(y)]$

$= [f(x) - f(a)]\big[g'\big(f(a)\big) + \gamma\big(f(x)\big)\big].$

$= (x - a)[f'(a) + \eta(x)][g'(b) + \gamma(x)].$

If $x \neq a$, we get

$\frac{h(x)-h(a)}{x-a} = [g'(b) + \gamma(y)][f'(x) + \eta(x)].$

If we take the limit as $x \to a$ and note that by theorem 1, f is continuous at a, we see that $y = f(x) \to f(a) = b$.

Hence, $\eta(x) \to 0$ and $\gamma(y) \to 0$. Therefore,

$$h'(a) = lim_{x \to a} \frac{h(x) - h(a)}{x - a} = g'(b)f'(a).$$

This complete the proof of the chain rule on differentiation.

The following theorem known as Inverse function theorem gives the relationship between the derivatives of inverse functions. If f is $1-1$ function on $[a, b]$, then $q[f(x)] = x$ $(a < x < b)$ where q is the inverse function for f.

7.5 INVERSE FUNCTION THEOREM

Theorem 7. (The Inverse Function Theorem). Let f be a $1-1$ real valued function on I. Let q be its inverse function. If f is continuous at $a \in I$, and q has the derivative at $b = f(a)$ with $q'(b) \neq 0$, then $f'(a)$ exists and $f'(a) = \frac{1}{q'(b)}$.

NOTES

Proof. For $h \neq 0$, let $v(h) = f(a+h) - f(a)$. Since f is 1-1, $v(h) \neq 0$ for $h \neq 0$. From this, we get

$$b + v(h) = f(a) + v(h) = f(a+h).$$

Hence, we have $q[b + v(h)] = q[f(a+h)] = a + h$, since q is the inverse function of f.

Now we have $\frac{f(a+h)-f(a)}{h} = \frac{[b+v(h)]-b}{a+h-a}$

$$= \frac{v(h)}{q[b+v(h)]-q(b)}$$

$$= \frac{1}{[q[b+v(h)]-q(b)]/v(h)} \quad .....(1)$$

By hypothesis f is continuous at a. So $\lim_{h\to 0} v(h) = 0$.

Thus, when $h \to 0$, the right side of (1) tends to the limit $\frac{1}{q'(b)}$. Hence we have

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \frac{1}{q'(b)}.$$

This completes the proof of the theorem.

Example 8. Using the Inverse Function Theorem, find the derivative of $f(x) = x^{1/n}$ where $n \in \mathbb{N}$ and $x > 0$.

The inverse function of f is x^n. Let $q(x) = x^n$. At any point $a > 0$, $q'(a) = na^{n-1}$. Hence, by the theorem

$$f'(a) = \frac{1}{q'(f(a))} = \frac{1}{n.a^{(n-1)/n}} = \frac{1}{n}.a^{(1/n)-1.}$$

7.6 DAURBOUX'S THEOREM ON DERIVATIVES

Theorem 9. (darboux property). If f has a derivative at every point of the closed interval $[a,b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.

NOTES

Proof. It is enough if we consider the case in which $f'(a) < f'(b)$. Thus if $f'(a) < k < f'(b)$, we have to show that there exists a c in (a, b) such that $f'(c) = k$.

Let us now define the function g on $[a, b]$ as$g(x) = f(x) - kx$ for $a \leq x \leq b$. From this we have as $g'(x) = f'(x) - k$ for $a \leq x \leq b$.

Thus $g'(x)$ exists for all x in $[a, b]$ and g is continuous on $[a, b]$ by theorem 1. Hence, by Theorem 8, g takes a minimum value at some point $c \in [a, b]$. But $g'(a) = f'(a) - k < 0$. Since $g'(x) < 0$ at $x = a$, g cannot attain its minimum value at $x = a$. For the minimum value of g at a, $g'(a) = 0$ by theorem 6. Similarly $g'(b) = f'(b) - k > 0$. So g cannot attain its minimum value at $x = b$ also. Thus we have $a < c < b$. We have $f'(c) = k$ which proves the theorem.

Example 10. Let $f(x) = 0$ for $-1 \leq x \leq 0$ and $f(x) = 1$ for $0 < x \leq 1$. Is there a function F such that $F'(x) = f(x)$ in [-1,1] ?

Suppose there exists a function F such that $F'(x) = f(x)$ in [-1,1]. Then, since $f(x)$ is defined in [-1,1], F' exists at every point of $[-1,1]$. So by the above Darboux property, F' takes every value between $F'(-1)$ and $F'(1)$. But $F'(1) = f(-1) = 0$ and $F'(1) = f(1) = 1$. So F' take every value between 0 and 1. But this cannot happen, since F' takes only two values 0 and 1 from the definition of F' in [-1,1]. So there is no function satisfied the given condition.

NOTES

CHECK YOUR PROGRESS

1. State inverse limit theorem.
2. State Darboux property.

7.7 ANSWER TO CHECK YOUR PROGRESS QUESTIONS

1. Let f be a $1-1$ real valued function on I. Let q be its inverse function. If f is continuous at $a \in I$, and q has the derivative at $b = f(a)$ with $q'(b) \neq 0$, then $f'(a)$ exists and $f'(a) = \frac{1}{q'(b)}$.
2. If f has a derivative at every point of the closed interval $[a, b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.

7.8 SUMMARY

15. If the real valued function f is differentiable at the point $a \in \mathbb{R}$, then f is continuous at a.
16. If f is differentiable at a, and g is differentiable at $f(a)$, then $h = f \circ g$ is differentiable at a and has the derivative $h'(a) = g'[f(a)]f'(a)$.

7.9 KEYWORDS

9. **Derivative:** Let f be a be a real valued function defined on an interval $I \subset \mathbb{R}$. If $a \in I$, then f is said to have a derivative at $x = a$, if $\lim_{x \to a} \frac{f(x)-f(a)}{h}$ exists. If this limit exists, then f is said to be differentiable at a and its derivative is denoted by $f'(a)$. Note that this limit is a real number. If we make the substitution $h = x - a$, then the above limit can also be written as $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

7.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. If $f(x) = |x^3|$ for $x \in \mathbb{R}$, find $f'(x)$ and $f''(x)$.
2. If $f(x) = x|x|$, prove that $f''(x) = 2$ if $x > 0$, and $f''(x) = -2$ if $x < 0$.
3. Suppose f is differentiable at $a \in I$ and $f'(a) \neq 0$. Prove that $|f|$ is differentiable at a and find $|f|'(a)$.

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4. If f is a function such that f^2 is derivable at a, dose it follow that f is derivable at a?

5. Show that the function f defined by $f(x) = x^2 \cos\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.

7.11 FURTHER READINGS

1. Arumugam & Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2010.
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UNIT-8 POWER SERIES

NOTES

Structure

8.0 INTRODUCTION

From elementary calculus, we know that the derivative of a real-valued function f on I at c gives the slope of the tangent to the curve $y = f(x)$ at $x = c$. Let f have derivatives at all points of I. Then $y = f(x)$ has tangents at all points of I. If $f'(c)$ exists, then the curve is said to be smooth at $x = c$. We have already noted in the previous section that if a real valued continuous function defined on the closed and bounded intervals attains the extremum value at c where $c \in I$ and $f'(c)$ exists, then $f'(c) = 0$. If the curve $y = f(x)$ has its end points on the x-axis (the curve crosses the x-axis at both the end points of the interval) and if it is smooth, it is geometrically evident that there will be horizontal tangent at some point on the curve. That is at some point on the curve, f' will become zero. This result is made precise in the following theorem known as Rolle's Theorem which is an important result in the differential calculus. We shall discuss in detail even the slight variation of the theorem so that readers will become familiar with all the aspects of this basic theorem in differential calculus.

8.1 OBJECTIVES

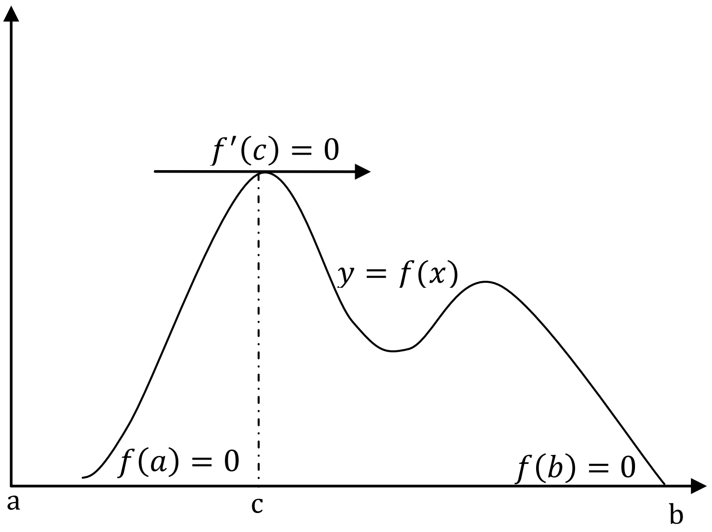
After going through this unit, you will be able to:

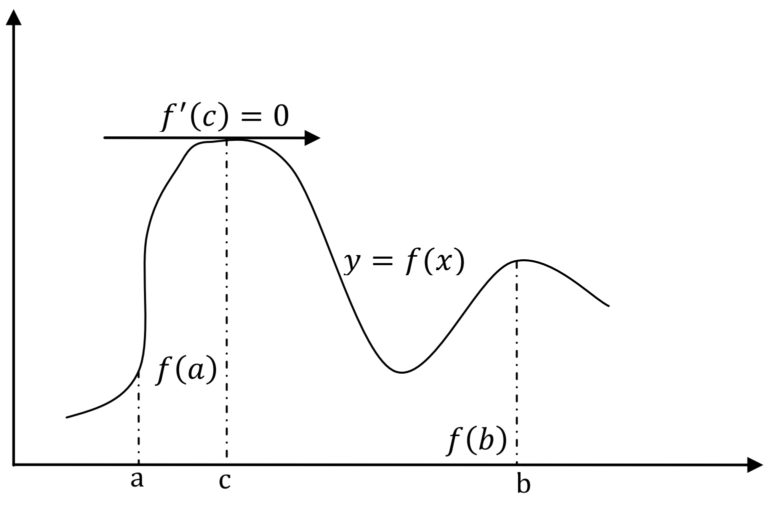
NOTES

- Understand what is meant by Rolle's Theorem, Mean value theorem
- Discuss the Fundamental theorem of calculus
- Discuss the properties of Power series expansion

8.2 ROLLE'S THEOREM

Theorem 1. $(Roll's\ Theorem)$ If f is continuous real-valued function defined on a bounded and closed interval $[a,b]$ with $f(a) = f(b) = 0$





and differentiable at every point x in the open interval (a,b) such that $f'(c) = 0$.

Proof. If f is identically zero on $[a,b]$, it attains a maximum and a minimum valu on $[a,b]$. If $f(x) > 0$ for some x in (a,b), the maximum value of f on $[a,b]$ will not be attained at a or$b$, sincev

NOTES

$f(a) = f(b) = 0$ by hypothesis. Hence, f will attain its maximum value will be attained in (a, b) so that for a point c in (a, b),$f'(c) = 0$.

Corollary. If $F(x)$ is a polynomial, then between any two roots of $F(x) = 0$,there exists at least one root of $F'(x) = 0$.

Proof. Let a and b be two roots of $F(x)$. Since $F(x)$ is a polynomial, $F(x)$ is continuous in $[a, b]$ and derivable at every point of (a, b). Since a and b are the roots of $F(x)$, we get $F(a) = F(b) = 0$. Hence, by Rolle's theorem, there exists at least one point c in (a, b) such that $F'(c) = 0$. This means that there exists at least one root of $F'(x) = 0$.

Note 1. The following statement of Rolle's theorem is an alternative form where we do not assume that f vanishes at $x = a$ and $x = b$. We shall give an independent proof of the theorem.

Theorem 2. If a function f is continuous in $[a, b]$ with $f(a) = f(b)$ and if f is differentiable at every point of (a, b), then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Proof. If f is constant throughout $[a, b]$, then f' is zero at all points of (a, b) so that the theorem is true.

Let f be not constant throughout $[a, b]$. Since f is continuous in $[a, b]$, it is bounded in $[a, b]$ and it attains one or other of its bounds in $[a, b]$ which is different from $f(a)$. Let U be its upper bound. Then f attains its upper bound at least once in (a, b). Then for all values x in $[a, b]$, $f(x) < f(c)$. So when h is an infinitesimal, we get

$f(c + h) - f(c) < 0.$

If h is positive, we get

$$\frac{f(c + h) - f(c)}{h} < 0$$

So that $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} < 0.$

The above inequality implies that $f'(c) < 0$ (1)

If h is negative, then we get

$$\frac{f(c + h) - f(c)}{h} < 0$$

NOTES

So that $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} < 0.$

The above inequality implies that $f'(c) > 0$ (2)

Combining (1) and (2), we get $f'(c) = 0$

Similarly we prove the theorem when f attains its lower bound in (a, b) which is different from $f(a)$.

Corollary. Theorem 1 can be deduced from Theorem 2.

Note 2. The above theorem can be stated in a slightly different form as follows.

Theorem 3. Let f be differentiable at every point of an open interval $(a, b)$and let f be continuous at both the end points a and b. If $f(a) = f(b)$, there is at least one point c in (a, b) at which $f'(c) = 0$.

Since f is differentiable at every point in the open interval(a, b), it is continuous in (a, b). By hypothesis, it is continuous at both the end points a and b. So f is continuous in the bounded closed interval $[a, b]$. Further note that we do not assume $f(a) = f(b) = 0$. It is enough modified for this slightly different form of the theorem, we shall give a different proof due to its importance.

Proof. Under the hypothesis, we shall assume that f' is never 0 in (a, b) and arrive at a contradiction. Since f is continuous on the bounded and closed interval $[a, b]$, it attains its maximum M and its minimum m at some points in $[a, b]$. None of the extreme values are attained at a point of (a, b). For if it attains the extreme values at the points in $(a, b), f'$ would vanish in (a, b) which is against our assumption. So they are both attained at the end points. Since $f(a) = f(b)$, then $m = M$ and hence f is constant on $[a, b]$ so that f' is zero on $[a, b]$. This contradicts our assumption that f' is never zero on $[a, b]$. Hence, for some point c in $(a, b), f'(c) = 0$.

Note. None of the conditions in the Rolle's theorem can be relaxed as shown by the following Examples 1 to 3

Example 1. Let $g(x) = x$ if $x \in [0,1)$ and $g(x) = 0$ if $x = 1$.

g is differentiable in the open interval and g(0) = g(1) = 0. It is continuous in the open interval (0,1) but not in the closed interval [0,1], since it is not left continuous at the right end point $x = 1$. Since g'$(x) = 1$ for every $x \in (0,1)$, there is no point c in (0,1) with

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g'$(c) = 0$. This shows that the loss of continuity at an end point is enough for the failure of the Rolle's theorem.

Example 2. let $f(x) = 1 - |x|$ for $x \in [-1,1]$. Now $f(-1) = f(1) = 0$ and f is a continuous function on $[-1,1]$. Since f'_+ and f'_- are different at $x = 0$, f obeys all differentiable at $x = 0$. So f is not differentiable in $(-1,1)$. Thus f obeys all the hypothesis of Rolle's theorem except that it is not differentiable at $x = 0$. For this f, there is no point c in $(-1,1)$ for which $f'(c) = 0$. Hence, the conclusion of Rolle's theorem is not true, if we weaken the assumption of the open interval.

Example 3. Let $f(x) = x$ in $[0,1]$. Then $f(x)$ is continuous in $[0,1]$ and derivable in $(0,1)$. But $f(0) \neq f(1)$. There is no point c in $[0,1]$ such that $f'(c) = 0$.

Example 4. Verify whether the function $f(x) = \sin x$ in $[0, \pi]$ satisfies the conditions of Rolle's theorem and hence find c as prescribed by the theorem.

We know that the function given by $f(x) = \sin x$ in $[0, \pi]$ is continuous in $[0, \pi]$ and differentiable everywhere in$(0, \pi)$. Further $f(0) = 0 = f(\pi)$. Hence, f satisfies the conditions of the Rolle's theorem. At $\pi/2 \in [0, \pi]$, $f'(x) = \cos x = 0$. So f' vanishes at $\pi/2 \in [0, \pi]$. Hence, in the Rolle's theorem $c = \pi/2$.

Example 5. Verify the Rolle's theorem for the function $f(x) = \sqrt{1 - x^2}\ (-1 < x < 1)$.

It is to be noted that f is continuous in the closed interval $[-1,1]$ and it is not derivable in $[-1,1]$, since this function has no derivatives at $x = -1$ or 1. It is differentiable in$(-1,1)$. Now if $c = 0$,we get $f'(0) = 0$. Thus the Rolle's theorem is true.

Example 6. Prove that there is no value of k such that the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0,1]$.

Let us suppose on the contrary that there is a real number k' such that $x^3 - 3x + k' = 0$ has α and β as its distinct roots in $[0,1]$ where $\alpha < \beta$ and $\alpha \neq \beta$. Now $\alpha \neq \beta$ and $\alpha, \beta \in [0,1]$ implies $0 < \alpha < 1$ and $0 < \beta < 1$. Since α and β are the roots of the above equation, we have $\alpha^3 - 3\alpha + k' = 0$, $\beta - 3\beta + k' = 0$. Now consider the function $f(x)$ defined in $[\alpha, \beta]$ as follows:

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$$f(x) = x^3 - 3x + k' \text{ for } x \in (\alpha, \beta).$$

Since $f(x)$ is a polynomial in x of degree 3, it is continuous on $[\alpha, \beta]$ and it is derivable in (α, β) with $f(\alpha) = f(\beta) = 0$. Hence, all the conditions of Rolle's theorem are satisfied by $f(x)$ in $[\alpha, \beta]$. Therefore, there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$. This implies $3c^2 - 3 = 0$. Hence, $c = \pm 1 \notin (0,1)$. This implies $c \in (\alpha, \beta)$ which contradicts our assumption that $0 < \alpha < 1$ and $0 < \beta < 1$. Therefore, there exists no real number k for which the given equation has two distinct roots in [0,1].

Example 7. Prove that if $\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$ has at least one root between 0 and 1.

Now consider the function defined by

$$f(x) = a_0 \frac{x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \cdots + a_n x \text{ in } [0,1].$$

Then $f(1) = 0$ by hypothesis and $f(0) = 0$. So $f(1) = 0 = f(0)$. Since f is a polynomial of degree n in [0,1], it is continuous and differentiable in [0,1]. Therefore , the hypothesis of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for some $x \in (0,1)$. So $f'(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $x \in (0,1)$.

We have already shown in the Example 7 of 7.1, that a function can have a derivative at each point of the interval but the derivative considered as a function need not be continuous. The following theorem shows that although they are not necessarily continuous, and derivatives like continuous functions satisfy the intermediate value property.

Theorem 4. (Darboux Property). If f has derivative at every point of the closed interval $[a, b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.

Proof. It is enough if we consider the case in which $f'(a) < f'(b)$. Thus if $f'(a) < k < f'(b)$, we have to show that there exists a c in (a, b) such that $f'(c) = k$. Let us now define the function g on $[a, b]$ as $g(x) = f(x) - kx$ for $a \le x \le b$. From this we have $g'(x) = f'(x) - k$ for $a \le x \le b$. Thus $g'(x)$ exists for all x in $[a, b]$ and g is continuous on $[a, b]$ by Theorem 1 of 7.1. Hence, by Theorem 8 of 4.5, g takes a minimum value at some point $c \in [a, b]$. But $g'(a) = f'(a) - k < 0$. Since $g'(x) < 0$ at $x = a$, g cannot attain its minimum value at $x = a$. For the minimum value of g at a, $g'(a) = 0$ by Theorem 6 of 7.1. similarly $g'(b) = f'(b) - k > 0$. So g cannot attain

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its minimum value at $x = b$ also. Thus we have $a < c < b$. We have $f'(c) = k$ which proves the theorem.

We shall illustrate the use of above theorem by the following example.

Example 8. Let $f(x) = 0$ for $-1 \leq x \leq 0$ and $f(x) = 1$ for $0 < x < 1$. Is there a function F such that

$F'(x) = f(x)$ in $[-1,1]$?

Suppose there exists a function F such that $F'(x) = f(x)$ in $[-1,1]$.

Then, since $f(x)$ is defined in $[-1,1]$, F' exists at every point of $[-1,1]$. So by the above

Darboux property, F' takes every value between $F'(-1)$ and $F'(1)$. But $F'(-1) = f(-1) = 0$ and

$F'(1) = f(1) = 1$.

So F' should take every value between 0 and 1. But this cannot happen, since F' takes only two values 0 and 1 from the definition of F' in $[-1,1]$. So there is no function F satisfying the given condition.

In the above example, it is important to note that f is not a continuous function in $[-1,1]$ and f is not the derivative of any function F in $[-1,1]$. But it will be shown later that if f is a continuous function on $[a, b]$, there can exists a function F on $[a, b]$, such that

$F'(x) = f(x)$ for all x in $[a, b]$.

8.3 MEAN VALUE THEOREM FOR DERIVATIVES

If we consider a smooth curve $y = f(x)$ in $[a, b]$, it is intuitively clear that at some point c in (a, b), the Slope of the tangent $f'(c)$ at $x = c$ will be equal to the slope of the chord joining the points a and b on the curve. This leads to the following theorem known as the mean value theorem for derivatives.

The most important aspect of the mean value theorem for derivatives is that it gives a relation between the derivative and the function so that we can obtain information about the function from

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the properties of the derivatives. We shall use Rolle's theorem to prove the following mean value theorem.

Theorem 1. (Mean Value Theorem for Derivatives). If f is a continuous function on the closed and bounded interval $[a,b]$ and if $f'(x)$ exists for all x in the open interval (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let us consider the function h defined as follows:

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

when $a \leq x \leq b$. From the definition of h, we have $h(a) = 0 = h(b)$. Further h is continuous in the closed interval $[a,b]$ and differentiable in the open interval (a,b). So h satisfies all the conditions of the Rolle's Theorem. Hence, there exists a point c in the open interval (a,b) such that $h'(c) = 0$.

But $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$ which proves that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

The above theorem is also known as Lagrange's Mean Value Theorem. It is important to note that the theorems do not exactly locate the positions of the points like c where the function takes one or more mean values. But what all it asserts is that the point lies between a and b. For some functions, the position of the point c may be specified well, but in most of the case it is very difficult to determine these points.

Note. The conclusion of the theorem may fail to be true if there is any point between a and b where the derivative of the function does not exist as shown by the following example.

Example 1. Let $f(x) = |x|$. This function is continuous everywhere on the real axis and has derivatives at all points of the real axis except at $x = 0$. Now consider the interval $[-1,2]$, $f(a) = f(-1) = 1$, $f(b) = f(2) = 2$.

Hence, $\frac{f(b)-f(a)}{b-a} = \frac{2-1}{3} = \frac{1}{3}$.

But $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$.

Example 2. Verify the hypothesis and the conclusion of the Mean Value Theorem for the following functions:

(i). $f(x) = \log x$ in $[1, e]$

(ii). $f(x) = Ax^2 + Bx + C$ in $[a, b]$

(iii). $f(x) = \frac{x}{x-1}$ in $2 < x < 4$

(i) The function $f(x) = \log x$ is continuous in $[1, e]$ and it has a derivative $f'(x) = \frac{1}{x}$ in $(1, e)$ and $f(e) - f(1) = \log e$. So the Mean Value Theorem implies $\log e = (e-1)\frac{1}{x}$ for a suitable x in $[1, e]$.

(ii) Being a polynomial, $Ax^2 + Bx + C$ is continuous in $[a, b]$ and derivable in (a, b).

$f(b) - f(a) = A(b^2 - a^2) + B(b - a)$ and $f'(x) = 2Ax + B$.

The Mean Value Theorem implies the existence of c in (a, b) such that

$$A(b^2 - a^2) + B(b - a) = [2Ac + B]\,(b - a).$$

Since $a \neq b$, we get $A(b + a) + B = 2Ac + B$.

Since $A \neq 0$, we get from the above $c = \frac{b+a}{2}$ which is in the open interval (a, b).

(iii) $F(x) = \frac{x}{x-1}$ is continuous in $[2,4]$ and differentiable in $(2,4)$,

$$f(4) - f(2) = -\frac{2}{3}, f'(x) = -\frac{1}{(x-1)^2}.$$

Hence, the existence of c in the Mean Theorem implies

$$-\frac{2}{3} = -\frac{2}{(c-1)^2} \text{ and hence } (c-1)^2 = 3.$$

Solving $(c-1)^2 = 3$ for c, we get $c = 1 \pm \sqrt{3}$.

Clearly, $c = 1 \pm \sqrt{3}$ lies in $[2,4]$, while $1 - \sqrt{3}$ does not belong to $[2,4]$. Hence $c = 1 + \sqrt{3}$.

The Mean Value Theorem can be expressed in the following alternative form.

Theorem 2. If a function $f(x)$ is continuous in the closed interval $[a, a + h]$ and differentiable in the open interval $(a, a + h)$ then there exists at least one number θ between 0 and 1 such that $f(a + h) - f(a) = hf'(a + \theta h)$.

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Proof. Let us take $b = a + h$ in the Mean Value Theorem. Then $a + \theta h$ is equal to a or b according as $\theta = 0$ or $\theta = 1$. Here if $0 < \theta < 1$, $a + \theta h$ is some point in (a, b). So c can be taken as $a + \theta h$ in the statement of the Mean Value Theorem. Hence, we obtain

$$f(a + h) - f(a) = hf'(a + \theta h).$$

Example 3. Determine θ that appears in the Mean Value Theorem given above for the function

$$f(x) = x^2 - 2x + 3 \text{ for } a = \frac{1}{2} \text{ and } h = \frac{1}{2}.$$

Now $f(a + h) - f(a) = -\frac{1}{4}$ and $hf'(a + \theta h) = (\frac{1}{2} + \frac{\theta}{2} - 1)$.

Hence, the Mean Value Theorem given above yields,

$$-\frac{1}{4} = \frac{1}{2} + \frac{\theta}{2} - 1 \text{ or } = \frac{1}{2}.$$

The following theorems are the very important consequences of the Mean Value Theorem.

Theorem 3. If f is a real valued function defined on closed interval $[a, b]$, such that $f'(x) = 0$ for all x in the open interval (a, b), then $f(x)$ must be a constant in the open interval (a, b).

Proof. Let x_1 and x_2 be any two points in (a, b) with $x_1 < x_2$, f satisfies all the conditions of the Mean Value Theorem in the $[x_1, x_2]$. Hence, by the Mean Value Theorem, there exists a point c in the open interval (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But $f'(c) = 0$ by hypothesis. Therefore, $f(x_1) - f(x_2) = 0$ for all x_1, x_2 in the open interval (a, b). Thus $f(x_1) = f(x_2)$ for any two different points x_1 and x_2 of (a, b). In other words, f is constant on (a, b).

Theorem 4. Let f and g be any two real valued differentiable functions on $[a, b]$ such that $f'(x) =$g'(x) for all x in $[a, b]$. Then $f(x)$ −g$(x) = c$ which is constant in $[a, b]$.

Proof. Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all x in $[a, b]$. Therefore, by the previous theorem $h(x) = c$.

Theorem 5. If f is a continuous real valued function on I and if $f'(x) > 0$ for all x in I except possibly at the end points of I, then f is strictly increasing on I and hence f is one-to-one.

Proof. Let us suppose $x_1, x_2 \in I$ with $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and is differentiable in (x_1, x_2). Then by the Mean Value Theorem, there is a point c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 > 0$ and $f'(c) > 0$ by hypothesis, it follows that $f(x_2) - f(x_1) > 0$. That is, $f(x_1) < f(x_2)$. Hence, f is strictly increasing on (a, b). So f is one-to-one.

A similar result holds good when $f'(x) < 0$ on I and we can state the result without proof as follows, since the proof runs parallel to the above theorem.

If f is differentiable on I and $f'(x) < 0$ for every $x \in I$, except possibly at the end points of I, then f is monotonic decreasing on I.

Example 4. Find the intervals in which the polynomial $2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

Let us take $f(x) = 2x^3 - 15x^2 + 36x + 1$.

Hence, $f'(x) = 6x^2 - 30x + 36 = 6(x - 2)(x - 3)$.

Now $f'(x) > 0$ for $x < 2$ and $x > 3$.

Further $f'(x) < 0$ for $2 < x < 3$

and $f'(x) = 0$ for $x = 2$ and $x = 3$.

Thus $f'(x)$ is positive in $(-\infty, 2)$ and $(3, \infty)$ and negative in $(2,3)$. Hence, f is monotonically increasing in the intervals $(-\infty, 2]$, $[3, \infty)$ and monotonically decreasing in $(2,3)$.

Theorem 6. If f' exists and is bounded on some interval I, then f is uniformly continuous on I.

Proof. Since f' is bounded in the I, there exists a $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. Then by

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applying the Mean Value Theorem to f in $[x_1, x_2]$, there exists $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

Consequently we get from the above hypothesis

$$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right| \leq M.$$

Hence, we obtain from the above inequality $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$. Since x_1 and x_2 are arbitrary points of I, it follows that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1| \text{ for all } x_1, x_2 \in I.$$

To show that f is uniformly continuous on I, let $\varepsilon > 0$ be given. Then we can choose a $\delta = \frac{\varepsilon}{M}$. Hence, if $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta$, we get $|f(x_2) - f(x_1)| < M|x_2 - x_1| < M\delta < \varepsilon$, the same δ serving for all points in I. Hence, f is uniformly continuous on I.

Example 5. Prove that the following functions are uniformly continuous:

(i) $f(x) = \frac{x}{x+2}$ in $[0,2]$.

(ii) $f(x) = x^2 + 4x$ in $[-1,1]$.

(i) For the function f in $[0,2]$, f' exists and $|f'(x)| < \frac{1}{2}$ which implies that f' is bounded in $[0,2]$. So by the above theorem f is uniformly continuous in $[0,2]$.

(ii) For the function $f$in $[-1,1]$, f' exists and $|f'(x) < 6|$ which implies that f' is bounded in $[-1,1]$. So by the above theorem f is uniformly continuous in $[-1,1]$.

Now we shall give below a generalization of the Mean Value Theorem known as Cauchy's Mean Value Theorem.

Theorem 7. Let f and g be continuous functions on the closed and bounded interval $[a, b]$ with $g(a) \neq g(b)$. If both f and g are differentiable at each point of the open interval (a, b) and $f'(x)$ and $g'(x)$ are not both equal to zero for any $x \in (a, b)$, then there exists a point c in the open interval (a, b), such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. Let us consider the function $h(x)$ defined as follows:

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$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x) - g(a)].$

Then $h(a) = 0$ and $h(b) = 0$. Using the hypothesis, $h(x)$ is continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b). So h satisfies all the hypotheses of the Rolle's theorem. Hence, there exists a point $c \in (a, b)$ such that $h'(c) = 0$. That is

$$f'^{(c)} - \frac{f(b)-f(a)}{g(b)-g(a)} \text{g}'(c) = 0.$$

If $\text{g}'(c)$ were zero, then $f'(c)$ would be zero, contradicting the hypothesis. Hence $\text{g}'(c) \neq 0$ so that we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Note 1. By taking $\text{g}(x) = x$ in the above theorem, we obtain the Lagrange's Mean Value Theorem as given in Theorem 1.

The generalized Mean Value Theorem proved above can be given in a slightly different form as follows.

Theorem 8. If f and g are each continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a c in the open interval (a, b) such that

$$f'(c)\,[\text{g}(b) - \text{g}(a)] = \text{g}'(c)[f(b) - f(a)].$$

Proof. Let $F(x) = f(x)[\text{g}(b) - \text{g}(a)] - \text{g}(x)\,[f(b) - f(a)]$. Then F is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Further, we can check easily that $F(a) = F(b)$. Hence, by Rolle's theorem, there is a point c in the open interval (a, b) with $F'(c) = 0$. But $F'(x) = f'(x)[\text{g}(b) - \text{g}(a)] - \text{g}'(x)[f(b) - f(a)]$. Hence

$$f'(c)[\text{g}(b) - \text{g}(a)] = \text{g}'(c)[f(b) - f(a)]$$

If $\text{g}(a) \neq \text{g}(b)$ and $\text{g}'(x) \neq 0$ and $f'(x) \neq 0$ for all $x \in (a, b)$, we can write the above expression in the form

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Example 6. Find the value of c in the generalized Mean Value Theorem for the following pairs of functions:

(i) $f(x) = \sqrt{x}$, $\text{g}(x) = 2x + 1$ in $[1,4]$.

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(ii) $f(x) = \sin x$, $g(x) = \cos x$ in $[-\frac{\pi}{2}, 0]$.

(i) $f(b) - f(a) = 1$, $g(b) - g(a) = 6$

and $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = 2$.

Hence, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}$ gives $\frac{1}{6} = \frac{1}{4\sqrt{x}}$.

Hence, we get $x = \frac{9}{4}$. Since $\frac{9}{4} \in (1,4)$, we get $c = \frac{9}{4}$.

(ii) the given functions satisfy all the conditions of the generalized Mean Value Theorem. Hence, we shall find the value of c as follows:

$$f(b) - f(a) = 1 \text{ and } g(b) - g(a) = 1.$$

Also $\frac{f'(x)}{g'(x)} = -\cos x$. Using these we get, from

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}, \cot x = -1.$$

The solution of this equation in $[-\frac{\pi}{2}, 0]$ is $\frac{\pi}{2}$. so c should be equal to $-\frac{\pi}{4}$.

8.4 TAYLOR'S THEOREM AND TAYLOR SERIES

Taylor's theorem is an extension of the Mean Value Theorem of differential calculus. To obtain the Taylor series, we have to consider the limit of the remainder after n terms in the Taylor's theorem. Though the different forms of the remainders can be obtained from the generalized law of mean, we derive them in the integral form, since the integral forms are easy to derive and handle.

First we shall give a brief motivation leading to the definition of Taylor series. Suppose for all x in some interval I, the function f can be expressed in the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \qquad (1)$$

where $a \in I$. Then we say that f is expanded in powers of $(x-a)$. Since the series on the right side of (1) expresses f, the question naturally arises whether the series on the right side of (1) converges to f. Also if it converges to f, let us find whether we can express $a_0, a_1, a_2, \ldots$ in terms of the properties of the function f. Assuming that (1) can be differentiated term by term with respect to x successively on both sides, we get

(i) $f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$

(ii) $f''(x) = 2a_2 + 2.3\, a_3(x-a) + 3.4a_4(x-a)^2 + \cdots$

and after nth differentiation, we get

$f^{(n)}(x) = n!\ a_n + (n+1)(n) \dots 2\, a_{n+1}(x-a) + (n+2)(n+1) \dots 3\, a_{n+2}(x-a)^2 + \cdots$ (iii)

If the substitution $x = a$ on both sides (i), (ii), and (iii) are permitted, then we get

$$a_0 = f(a), a_1 = f'(a), a_2 = 2!\ f''(a), \dots, f^{(n)}(a) = n!\, a_n$$

Using the above relations, (1) can be written as $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$ (2)

(2) is called the Taylor series of the function about $x = a$. When $a = 0$, we get from (2) the following expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}\, x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots \qquad (3)$$

The expansion (3) of at $x = 0$ is called the Maclaurin series for f.

Note. For the expansion of f at $x = a$, $f^{(n)}(a)$ must exist for all $n = 1, 2, 3, \dots$. The series may not converge for any x except $x = a$.

For investigating the Taylor series, first we shall consider the partial sum of Taylor series with remainder which tends to zero for large values of n yielding Taylor series. So we shall establish the following theorem known as Taylor's formula.

Theorem 1. (Taylor's Formula). Let f be a real valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}(x)$ is continuous on $[a, a+h]$. Then we have

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \text{ for } x \in [a, a+h]$$

where $R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n\ f^{(n+1)}(t)dt$.

Proof. First we shall establish that

if $R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n\ f^{(n+1)}(t)dt$, then

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$R_n(x) - R_{n+1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n$ for $x \in [a, a+h]$, $n = 1, 2, 3, \ldots$

To see this, we have by integration by parts

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n \ f^{(n+1)}(t)dt$$

$$= \left[\frac{(x-t)^n}{n!} f^{(n)}(t)\right]_{t=a}^{t=x} + \frac{1}{(n-1)!}\int_a^x (x-t)^{(n-1)} \ f^{(n)}(t)dt$$

$$= -\frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x).$$

It follows that

$$R_n(x) - R_{n+1}(x) = \frac{(x-a)^n}{n!} f^{(n)}(a) \qquad (1)$$

Using (1), we shall establish Taylor's formula.

From the definition of $R_n(x)$, we get

$$R_1(x) = \int_a^t f'(t)\, dt \ = \ f(x) - f(a)$$

Further from (1), we get

$$R_1(x) - R_2(x) = \frac{f'(a)}{1!}(x-a);$$

$$R_2(x) - R_3(x) = \frac{f''(a)}{2!}(x-a)^2.$$

Proceeding in this manner, we get

$$R_n(x) - R_{n+1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

Adding all the above equations, we obtain

$$-R_{n+1}(x) = -f(x) + f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

From the above we get

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

which completes the proof of the theorem.

Note. The Taylor series of f around $x = a$ need not converge in general to $f(x)$ at any point x in the neighbourhood of $x = a$. For example consider the function $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. We shall show in the next section that f has derivatives of all orders at every point in the neighbourhood of $x = 0$ and for all n, $f^{(n)}(0) = 0$. The Taylor series around 0 is given by

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

This series has sum zero. But the sum of the Taylor series at any other point in the neighbourhood of $x = 0$ is different from zero. Thus the series of the function does not converge to $f(x)$ for any point in the neighbourhood of $x = 0$ except at $x = 0$.

The following theorem gives the condition under which the Taylor series of f converges to $f(x)$.

Theorem 2. The Taylor series of f converges to $f(x)$ at $x = a$ if and only if $R_n(x) \to 0$ as $n \to \infty$.

Proof. From the Taylor's formula, we get

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n(x) \qquad (1)$$

where $R_n(x)$ is the remainder after n terms.

Let

$$s_n(x) = f(a) + + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) \qquad (2)$$

From (1) and (2), we get $f(x) = s_n(x) + R_n(x)$. The Taylor series converges to $f(x)$, then $s_n(x) \to f(x)$ as $n \to \infty$. This implies and is implied by $\lim_{n\to\infty} R_n(x) = 0$, when

$$s_n(x) = f(x) - R_n(x).$$

Note. Further the function in the note under theorem 1 cannot be represented by Taylor series about the origin. By Taylor's theorem, the expansion of the function about the origin,

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

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where $R_n(x) = e^{-\frac{1}{x^2}}$. Now $R_n(x)$ does not tend to zero as $n \to \infty$. Hence, $e^{-\frac{1}{x^2}}$ cannot be represented by Taylor series.

Since the convergence of the Taylor series depends upon the limit of the remainder term $R_{n+1}(x)$ as $n \to \infty$, we shall put the remainder term in two different convenient forms due to Lagrange and Cauchy.

Theorem 3. (Taylor's formula with Lagrange's form of the remainder)

Let f be a real valued function on $[a, a+h]$ such that $f^{n+1}(x)$ exists for every $x \in [a, a+h]$ and f^{n+1} is continuous on $[a, a+h]$. Then if $x \in [a, a+h]$, there exists a number c with $a < c < x$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}.$$

The same result is true if $h < 0$. In that case, $[a, a+h]$ is replaced by $[a+h, a]$.

Proof. We shall apply the Second Mean Value Theorem of integral calculus in the integral form of remainder. Then

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

$$= \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n \, dt = \frac{f^{(n+1)}(c)}{n!} \frac{(x-t)^{n+1}}{(n+1)}$$

for some $c \in [a, x]$. Then we get from the above

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

which completes the proof.

Corollary 1. Let $x = a + h$, then $c = a + \theta h$, where $0 < \theta < 1$. Hence making these substitutions in the theorem, we get

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n)}(a)}{n!}h^n + \frac{f^{(n+1)}(a+\theta h)}{(n+1)!}h^{n+1}$$

Corollary 2. (Maclaurin's Theorem with Lagrange's Form of Remainder).

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$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.$$

Proof. Taking $a = 0$ and $h = x$ in Corollary 1 we obtain Maclaurin's theorem with Lagrange's form of remainder in $[0, x]$.

Theorem 4. (Taylor's Formula with Cauchy Form of Remainder).

Let f be a real valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}(x)$ is continuous on $[a, a+h]$. Then if $x \in [a, a+h]$ there exists a number c with $a \le c \le x$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)(x-c)^n}{(n+1)!}(x-c)^n(x-a).$$

The same result is true for $h < 0$ and $[a, a+h]$ is replaced by $[a+h, a]$.

Proof. Applying the second Mean Value Theorem of integral calculus taking the integrand as a whole, we get

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt = \frac{f^{(n+1)}(c)(x-c)^n}{n!}\int_a^x dx$$

for some $c \in [a, a+h]$, Thus we have

$$R_{n+1}(x) = \frac{f^{(n+1)}(x)(x-c)^n}{n!}.(x-a)$$

which completes the proof.

Corollary 1. Let $x = a+h$. Then $c = a + \theta h$ for some $0 < \theta < 1$. Hence , there exists θ with $0 < \theta < 1$, such that

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n)}(a)}{n!}h^n + \frac{f^{(n+1)}(a+\theta h)}{(n+1)!}(1-\theta)^n h^{n+1}.$$

Corollary 2. (Maclaurin's Theorem with Cauchy Form of Remainder).

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}(1-\theta)^n x^{n+1}.$$

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Proof. Taking $a = 0$ and $h = x$ in Corollary 1, we get Maclaurin's theorem with Cauchy's form of remainder in $[0, x]$.

Example: Write down Taylor formula with Lagrange form of Remainder for $f(x) = \log(1 + x)$ about $a = 2$ and $n = 4$.

The Taylor's theorem about $a = 2$ for $n = 4$ is

$$f(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \frac{f^{(5)}(c)}{5!}(x-2)^5$$

$f(x) = \log(1 + x)$, $f'(x) = \frac{1}{(1+x)}$, $f''(x) = -\frac{1}{(1+x)^2}$

$f^{(3)}(x) = \frac{2}{(1+x)^3}$, $f^{(4)}(x) = -\frac{6}{(1+x)^4}$, $f^{(5)}(x) = \frac{24}{(1+x)^5}$.

Hence the required expansion,

$$f(x) = \log 3 + \frac{1}{3}.\frac{(x-2)}{1!} - \frac{1}{9}.\frac{(x-2)^2}{2!} + \frac{2}{27}.\frac{(x-2)^3}{3!} - \frac{6}{81}.\frac{(x-2)^4}{4!} + \frac{1}{5!}.\frac{24}{(1+c)^5}(x-2)^5$$

where c is between 2 and x in $(-1, \infty)$.

8.5 POWER SERIES EXPANTION

Introduction

The terms of the series which we have examined so far (with the exception of those considered in the chapter on Uniform Convergence) were for the most part, determinate numbers. In such cases the series may be characterized at having constant terms. This, however, was not everywhere the case. In the geometric series$\sum r^n$, for instance, the terms only become determinate when the value of r is assigned. Our investigation of the behavior of this series did not terminate with the mere statement of the convergence or divergence, the result was: thee series converges if $|r| < 1$, but diverges if $|r| \geq 1$. The solution thus depends, as do the terms of the series, on the value of a quantity left undetermined a variable. In this chapter we propose only to consider, in detail within the scope of the present work, series whose *generic term* has the form $a_n x^n$, i.e., we shall consider series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \equiv \sum_{n=0}^{\infty} a_nx^n$$

Such series are called *power series* (in x) and the numbers a_n(dependent on n but not on x) their coefficients.

Definition.

For $x = 0$, obviously every power series is convergent whatever be the value of the coefficients. The most important fact about a power series is that either:

i. it converges for no value of x other than the self-evident point $x = 0$, we then say that it is *nowhere convergent*, e.g., $\sum n^n x^n$ converges for no value of x other than $x = 0$,
or

ii. it converges for all values of x, and is then called *everywhere convergent*,
e.g., $\sum \frac{x^n}{n!}$, $\sum (-1)^n \frac{x^n}{n!}$,
or

iii. (the general case) it converges for some values of x and diverges for others the totality of points x for which it converges is called its *region of convergence*.

Thus, if $\sum a_nx^n$ is a power series which does not converge everywhere or nowhere, then a definite positive number R exists such that $\sum a_nx^n$ converges (indeed absolutely) for every $|x| < R$ but diverges for every $|x| > R$. The number R, which is associated with every power series, is called the *radius of convergence* and the interval, $]-R, R[$, the *interval of convergence*, of the given power series.

The behavior of a power series at $|x| = R$, depends entirely upon the character of the sequence $\{a_n\}$ of its coefficients. For instance, both the series

$$\sum \frac{x^n}{n^2}, \sum \frac{x^n}{n}$$

converge when $|x| < 1$ and diverge when $|x| > 1$. When $|x| = 1$, the first series converges while the second diverges at $x = 1$, and converges at $x = -1$.

Properties of functions expressible as Power Series

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In the section, we shall derive some properties of the functions which can be expressed in terms of power series, $i.e.$, the functions of the form

$f(x) = \sum_{n=0}^{\infty} a_n x^n$, or $f(x) = \sum_{n=0}^{\infty} a^n (x-a)^n$,

the former being the power series expansion of $f(x)$ about the origin, while the latter is about $x = a$. This can, however, be thought of in the reverse direction also. In the interval of convergence, the power series $\sum a_n x^n$ or $\sum a_n (x-a)^n$ has a definite sum $f(x)$ for each x, and usually different sum for a different x. In order to express this dependence on x, we write

$f(x) = \sum a_n x^n$, or $f(x) = \sum a_n (x-a)^n$,

$f(x)$ is then called the sum function of the series.

Before proceeding to the next theorem, let us understand an important distinction between the intervals of absolute and of uniform convergence. An interval of uniform convergence must include its end points but the interval of absolute convergence need not.

Thus, if a power series converges absolute and uniformly for $|x| < R$, we express this fact by saying that it converges absolutely in $]-R, R[$, and uniformly in $[-R+\varepsilon, R-\varepsilon]$, no matter which $\varepsilon > 0$ is chosen; the latter interval may be replaced by $[-R_1, R_1], R_1 < R$.

Theorem 4. If a power series $\sum a_n x^n$ converges for $|x| < R$, and let us define a function

$$f(x) = \sum a_n x^n,\ |x| < R,$$

then $\sum a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$, no matter which $\varepsilon < 0$ is chosen, and that the function f is continuous and differentiable on $]-R, R[$, and

$$f'(x) = \sum n a_n x^{n-1},\ |x| < R$$

Let $\varepsilon < 0$ be any number given.

For $|x| \leq R - \varepsilon$, we have

$$|a_n x^n| \leq |a_n| (R-\varepsilon)^n.$$

But since $\sum a_n (R-\varepsilon)^n$, converges absolutely (every power series converges absolutely within its interval of convergence), therefore by Weierstrass's M-test, the series $\sum a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

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Again, since very term of the series $\sum a_n x^n$ is continuous and differentiable on $]-R, R[$, and $\sum a_n x^n$ is uniformly convergent on $[-R+\varepsilon, R-\varepsilon]$, therefore its sum function f is also continuous and differentiable on $]-R, R[$.

Also, $\overline{\lim}_{n\to\infty} |na_n|^{1/n} = \overline{\lim}_{n\to\infty} (n^{1/n}) |a_n|^{1/n} = 1/R$

Hence, the differentiated series $\sum na_n x^{n-1}$ is also a power series and has the same radius of convergence R as $\sum a_n x^n$. Therefore, $\sum na_n x^{n-1}$ is uniformly convergent in $[-R+\varepsilon, R-\varepsilon]$.

Hence, $f'(x) = \sum na_n x^{n-1}, |x| < R$.

Corollary. Under the hypothesis of the above theorem, f has derivatives of all orders in$]-R, R[$, which are given by

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\ldots(n-m+1)a_n x^{n-m}$$

and in particular,

$$f^{(m)}(0) = m!\, a_m, m = 0,1,2, \ldots$$

[Here, as usual, $f^{(m)}$ denotes the mth derivative of f for $m = 1,2,3, \ldots$, and $f^{(0)}$ means f.]

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

By the above theorem, $f(x)$ is differentiable any number of times. Let us differentiate m times.

$\therefore\ f^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + na_n x^{n-1} + \ldots$

$f^{(2)}(x) = 2.1a_2 x + 3.2a_3 x^2 + 4.3a_4 x^2 \ldots + n(n-1)a_n x^{n-2} + \ldots$

$f^{(3)}(x) = 3!\, a_3 + 4.3.2a_4 x + \ldots + n(n-1)(n-2)a_n x^{n-3} + \ldots$

$f^{(4)}(x) = 4!\, a_4 + \ldots + n(n-1)(n-2)(n-3)a_n x^{n-4} + \ldots$

.

.

$f^{(m)}(x) = m!\, a_m + (m+1)m(m-1) \quad \ldots 2.\, a_{m+1} x + \quad \ldots \quad + n(n-1)\ldots(n-m+1)a_n x^{n-m} + \ldots$

$= \sum_{n=m}^{\infty} n(n-1)\ldots(n-m+1)a_n x^{n-m}$

NOTES

Also

$$f^{(m)}(0) = m!\, a_m$$

the other terms vanishing at $x = 0$.

Now, it appears natural to pose the question whether the converse assertion is true. The problem can be stated follows.

Suppose a function $f(x)$ is infinitely differentiable on an interval $]-R, R[$, $R \neq 0$. We can formally construct the Taylor's series for this function:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Now does this series converge on the interval $]-R, R[$, and will its sum be equal to the function f in case it exists? It turns out that in general the answer to the question is negative which can be confirmed by the example of the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$$

In fact it can be easily verified that the function is infinitely differentiable through the x-axis and that at the origin , we have

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$$

Consequently, all the coefficients of the Taylor's series of the function are equal to zero. Thus the Taylor's series converges on the entire x-axis and its sum is identically equal to zero, whereas the function takes on a zero value only at the origin and so we fail to express f as a power series.

Abel's Theorem

In this section we shall prove that for a power series which has a given radius of convergence and convergent at an end-point of the interval, the interval of uniform convergence extends up to and includes that end-point. Moreover, in that case the sum function f is continuous not only in $]-R, R[$, but also at the end point. This is proved in Abel's Theorem:

Theorem. Abel's Theorem (First form).

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If a power series $\sum_{n=0}^{\infty} a_n\ x^n$ converges at the end point $x = R$ of the interval of convergence $]-R, R[$, then it is uniformly convergent in the closed interval $[0, R]$.

we shall show that under the assumptions of the theorem, Cauchy's criterion for uniform convergence is satisfied on the closed interval $[0, R]$. This will imply the uniform convergence of the series on $[0, R]$.

Let $S_{n,p} = a_{n+1} R^{n+1} + a_{n+2} R^{n+2} + \cdots + a_{n+p} R^{n+p},\ \ p = 1, 2, 3, \ldots$

Then obviously $\quad a_{n+1} R^{n+1} = S_{n,1}$

$$a_{n+2} R^{n+2} = S_{n,2} - S_{n,1}$$

$$\vdots$$

$$a_{n+p} R^{n+p} = S_{n,p} - S_{n,p-1}$$

Let $\varepsilon > 0$ be given.

Since the number $\sum_{n=0}^{\infty} a_n\ R^n$ is convergent , therefore by Cauchy's General Principle of convergence, there exists an integer N such that for $n \geq N$,

$$|S_{n,q}| < \varepsilon, \text{ for all } q = 1,2,3, \ldots \qquad (2)$$

Taking into account that

$$\left(\frac{x}{R}\right)^{n+p} \leq \left(\frac{x}{R}\right)^{n+p-1} \leq \cdots \leq \left(\frac{x}{R}\right)^{n+1} \leq 1, \text{ for } 0 \leq x \leq R$$

and using equations (1) and (2), we have for $n \geq N$

$$|a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \cdots + a_{n+p} x^{n+p}|$$

$$= \left|a_{n+1} R^{n+1} \left(\frac{x}{R}\right)^{n+1} + a_{n+2} R^{n+2} \left(\frac{x}{R}\right)^{n+2} + \cdots + a_{n+p} R^{n+p} \left(\frac{x}{R}\right)^{n+p}\right|$$

$$= |S_{n,1} \left\{\left(\frac{x}{R}\right)^{n+1} - \left(\frac{x}{R}\right)^{n+2}\right\} + S_{n,2} \left\{\left(\frac{x}{R}\right)^{n+2} - \left(\frac{x}{R}\right)^{n+3}\right\} + \cdots$$

$$+ S_{n,p-1} \left\{\left(\frac{x}{R}\right)^{n+p-1} - \left(\frac{x}{R}\right)^{n+p}\right\} + S_{n,p} \left(\frac{x}{R}\right)^{n+p} |$$

$$\leq |S_{n,1}| \left\{\left(\frac{x}{R}\right)^{n+1} - \left(\frac{x}{R}\right)^{n+2}\right\} + |S_{n,2}| \left\{\left(\frac{x}{R}\right)^{n+2} - \left(\frac{x}{R}\right)^{n+3}\right\} + \cdots$$

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$$< \varepsilon\left\{\left(\frac{x}{R}\right)^{n+1} - \left(\frac{x}{R}\right)^{n+2} + \left(\frac{x}{R}\right)^{n+2} - \left(\frac{x}{R}\right)^{n+3} + \cdots - \left(\frac{x}{R}\right)^{n+p} + \left(\frac{x}{R}\right)^{n+p}\right.$$

$$= \varepsilon\left(\frac{x}{R}\right)^{n+1} \leq \varepsilon.$$

for all $n \geq N, p \geq 1$ and for all $x \in [0, R]$.

Hence by Cauchy's criterion , the series converges uniformly on $[0, R]$.

Theorem . Abel's Theorem(Second form).

If $\sum a_n x^n$ be a power series with finite radius of convergence R, and let

$$f(x) = \sum a_n x^n, -R < x < R$$

If the series $\sum a_n R^n$ converges, then

$$\lim_{x \to R-0} f(x) = \sum a_n R^n$$

Let us first show that there is no loss of generality in taking $R = 1$.

Put $x = Ry$, so that

$$\sum a_n x^n = \sum a_n R^n y^n = \sum b_n y^n, \text{ where } b_n = a_n R^n.$$

It is a power series with radius of convergence R', where

$$R' = \frac{1}{\overline{\lim}|a_n R^n|^{1/n}} = 1$$

Thus, it will suffice to prove the following:

Let $\sum_0^\infty a_n x^n$ be a power series with unit radius of convergence and let

$$f(x) = \sum_0^\infty a_n x^n, \; -1 < x < 1$$

If the series $\sum a_n$ converges, then

$$\lim_{x \to 1-0} f(x) = \sum_0^\infty a_n$$

Let $S_n = a_0 + a_1 + a_2 + \cdots + a_n$, $S_{-1} = 0$, and let $\sum_{n=0}^\infty a_n = S$, then

$$\sum_{n=0}^m a_n x^n = \sum_{n=0}^m (S_n - S_{n-1}) x^n = \sum_{n=0}^{m-1} S_n x^n + S_m x^m - \sum_{n=0}^m S_{n-1} x^n$$

$= \sum_{n=0}^{m-1} S_n\, x^n - x \sum_{n=0}^{m} S_{n-1}\, x^{n-1} + S_m\, x^m = (1-x) \sum_{n=0}^{m-1} S_n\, x^n + S_m\, x^m.$

For $|x| < 1$, when $m \to \infty$, since $S_m \to S$, and $x^m \to 0$, we get

$$f(x) = (1-x) \sum_{n=0}^{\infty} S_n\, x^n, \text{ for } 0 < x < 1 \qquad (1)$$

Again, $S_n \to S$, for $\varepsilon > 0$, there exists N such that

$$|S_n - S| < \frac{\varepsilon}{2}, \text{ for all } n \geq N \qquad (2)$$

Also $$(1-x) \sum_{n=0}^{\infty} x^n = 1, \quad |x| < 1 \qquad (3)$$

Hence, for $n \geq N$, we have , for $0 < x < 1$,

$$|f(x) - S| = |(1-x) \sum_{n=0}^{\infty} S_n\, x^n - S| \qquad \text{[using (1)]}$$

$$= |(1-x) \sum_{n=0}^{\infty} (S_n - S)\, x^n| \qquad \text{[using (3)]}$$

$$\leq (1-x) \sum_{n=0}^{N} |S_n - S| x^n + \frac{\varepsilon}{2} (1-x) \sum_{n=N+1}^{\infty} x^n \qquad \text{[using (2)]}$$

$$\leq (1-x) \sum_{n=0}^{N} |S_n - S| x^n + \frac{\varepsilon}{2}$$

But for a fixed N, $(1-x) \sum_{n=0}^{N} |S_n - S| x^n$ is a positive function of x, having zero value at $x = 1$. Therefore, there exists $\delta > 0$, such that for $1 - \delta < x < 1$,

$$(1-x) \sum_{n=0}^{N} |S_n - S| x^n < \frac{\varepsilon}{2}.$$

$$\therefore \quad |f(x) - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ when } 1 - \delta < x < 1.$$

Hence, $\lim_{x \to 1-0} f(x) = S = \sum_{n=0}^{\infty} a_n$.

Corollary. If the series $\sum (-1)^n\, a_n$ converges, then

$$\lim_{x \to -1+0} f(x) = \sum (-1)^n\, a_n\, .$$

Putting $y = -x$, and $b_n = (-1)^n\, a_n$, we have

$\lim_{x \to -1+0} f(x) = \lim_{x \to -1+0} \sum a_n\ x^n =$
$\lim_{x \to -1+0} \sum (-1)^n a_n\ (-x)^n$

$$= \sum_{y \to 1-0} \sum b_n\, y^n = \sum b_n\, .$$

NOTES

Check your progress

1) State Roll's Theorem
2) State Darboux Property
3) State mean value theorem for derivatives
4) If f and g be any two real valued differentiable functions on $[a,b]$ such that $f'(x)$ =g'(x) for all x in $[a,b]$. Then? $f(x)$ −g$(x) = c$ which is constant in $[a,b]$.

8.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1) If f is continuous real-valued function defined on a bounded and closed interval $[a,b]$ with $f(a) = f(b) = 0$ and differentiable at every point x in the open interval (a,b) such that $f'(c) = 0$.

2) If f has derivative at every point of the closed interval $[a,b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.

3) If f is a continuous function on the closed and bounded interval $[a,b]$ and if $f'(x)$ exists for all x in the open interval (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

4) Let f and g be any two real valued differentiable functions on $[a,b]$ such that $f'(x)$ =g'(x) for all x in $[a,b]$. Then $f(x)$ −g$(x) = c$ which is constant in $[a,b]$.

8.7 SUMMARY

- **(Rolle's theorem)** If f is continuous real-valued function defined on a bounded and closed interval $[a,b]$ with $f(a) = f(b) = 0$ and differentiable at every point x in the open interval (a,b) such that $f'(c) = 0$.
- **(Darboux Property)** If f has derivative at every point of the closed interval $[a,b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.
- **(Mean value theorem for derivatives)** If f is a continuous function on the closed and bounded interval $[a,b]$ and if $f'(x)$ exists for all x in the open interval (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
- If f is a real valued function defined on closed interval $[a,b]$, such that $f'(x) = 0$ for all x in the open interval (a,b), then $f(x)$ must be a constant in the open interval (a,b).

NOTES

- If f and g are each continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a c in the open interval (a, b) such that
$f'(c)\,[g(b) - g(a)] = g'(c)[f(b) - f(a)]$.
- **(Taylor's Formula)** Let f be a real valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}(x)$ is continuous on $[a, a+h]$. Then we have
$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \text{ for } x \in [a, a+h]$$
where $R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n\, f^{(n+1)}(t)dt$.
The Taylor series of f converges to $f(x)$ at $x = a$ if and only if $R_n(x) \to 0$ as $n \to \infty$.
- **(Taylor's formula with Lagrange's form of the remainder)**
Let f be a real valued function on $[a, a+h]$ such that $f^{n+1}(x)$ exists for every $x \in [a, a+h]$ and f^{n+1} is continuous on $[a, a+h]$. Then if $x \in [a, a+h]$, there exists a number c with $a < c < x$ such that
$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}.$$
The same result is true if $h < 0$. In that case, $[a, a+h]$ is replaced by $[a+h, a]$.
- **(Taylor's Formula with Cauchy Form of Remainder).**
Let f be a real valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}(x)$ is continuous on $[a, a+h]$. Then if $x \in [a, a+h]$ there exists a number c with $a \le c \le x$ such that
$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)(x-c)^n}{(n+1)!}(x-c)^n(x-a).$$
The same result is true for $h < 0$ and $[a, a+h]$ is replaced by $[a+h, a]$.
- **Abel's Theorem (First form).**

If a power series $\sum_{n=0}^{\infty} a_n\, x^n$ converges at the end point $x = R$ of the interval of convergence $]-R, R[$, then it is uniformly convergent in the closed interval $[0, R]$.

- **Abel's Theorem(Second form).**
If $\sum a_n\, x^n$ be a power series with finite radius of convergence R, and let
$$f(x) = \sum a_n\, x^n, -R < x < R$$

NOTES

If the series $\sum a_n R^n$ converges, then

$$\lim_{x \to R-0} f(x) = \sum a_n R^n .$$

8.8 KEYWORDS

- **(Rolle's theorem)** If f is continuous real-valued function defined on a bounded and closed interval $[a, b]$ with $f(a) = f(b) = 0$ and differentiable at every point x in the open interval (a, b) such that $f'(c) = 0$.
- **(Darboux Property)** If f has derivative at every point of the closed interval $[a, b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.
- **(Mean value theorem for derivatives)** If f is a continuous function on the closed and bounded interval $[a, b]$ and if $f'(x)$ exists for all x in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

8.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

1) Give the geometrical interpretation of the mean value theorem.
2) Show that the real valued function $f(x) = x^2 + 2$ defined on $[1,3]$ is strictly increasing.
3) Show that $\frac{1}{x}\log(1 + x)$ decreases as x increases from 0 to ∞.
4) Show that $x^3 - 3x^2 + 2x + 3$ is monotonically increasing in every interval.

8.10 FURTHER READINGS

1) Arumugam & Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2010.
2) Richard R. Goldbrg, Methods of Real Analysis, Oxford & IBH Publishing Company, New Delhi.
3) D. Somasundaram & B. Choudhary, A first course in Mathematical Analysis, Narosa Publishing House, Chennai.
4) M.K. Singhal & Asha Rani Singhal, A First Course in Real Analysis, R. Chand & Co. June 1997 Edition.
5) Shanthi Narayan, A Couse of Mathematical Analysis, S. Chand & Co., 1995

BLOCK III
INTEGRAL FUNCTIONS AND CONTRACTION MAPPING THEOREM

NOTES

UNIT IX RIEMANN INTEGRATION

Structure

9.0 INTRODUCTION

The Riemann integration is a basic concept in mathematical analysis, since it relates to boundedness, continuity and differentiability. In this chapter, we shall give a detailed and rigorous account of Riemann integration, proving the basic property of integration as anti-derivative which comes out as the fundamental theorem of calculus. In this chapter, we shall consider only the real valued function.

9.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by Riemann Integral
- Discuss Daurbox's theorem
- Discuss the Conditions for Integrability

NOTES

9.2 DEFINITION OF THE RIEMANN INTERGRAL

Definition 1. Let I be a bounded and closed interval of . Let f be a bounded real valued function defined on I. Let us define the following

$$M[f; I] = \text{lub}_{x\in I} f(x), \ \ m[f; I] = \text{glb}_{x\in I} f(x).$$

Definition 2. A partition P of $[a, b]$ is a finite subset$\{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ such that P: $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

The points $x_0, x_1, x_2, \dots, x_n$ are called the points of sub-division of $[a, b]$. The closed interval

$$I_1 = [x_0, x_1], \qquad I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$

are called the component intervals of $[a, b]$. For the partition P, we have in the above notation

$$M[f; I] = \underset{x\in I}{\text{lub}} f(x), \ \ m[f; I] = \underset{x\in I}{\text{glb}} f(x)$$

where $k = 1, 2, 3, \dots, n$.

From the definition of partition we have

$$m[f; I] \leq m[f; I_k] \leq M[f; I_k] \leq M[f; I] \text{ for each } k.$$

Definition 3. Let f be a bounded function on the closed bounded interval $[a, b]$ and let P be any partition of $[a, b]$. We define the upper sum of f corresponding to the partition P as

$$U[f; P] = \sum_{k=1}^{n} M[f; I_k] \, |I_k|.$$

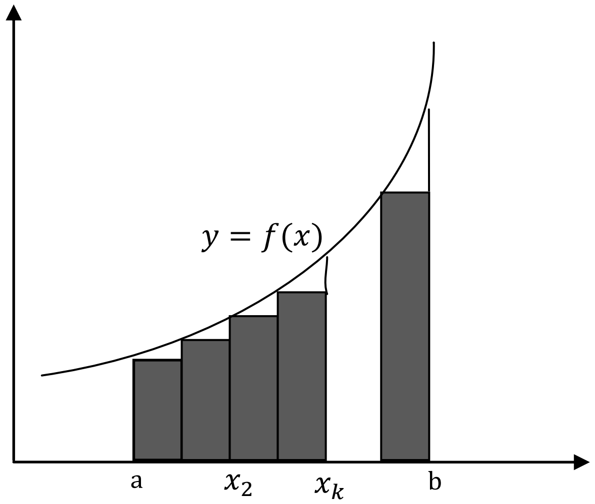
Similarly, the lower sum of f is defined as

$$L[f; P] = \sum_{k=1}^{n} m[f; I_k] \, |I_k|.$$

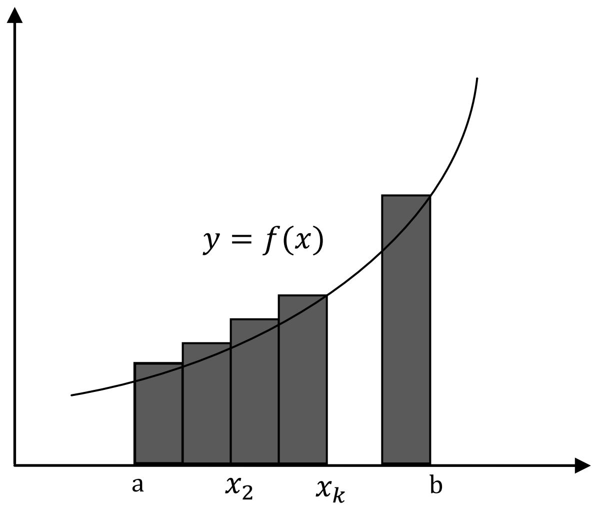
Since $m[f; I_k] < M[f; I_k]$ always, we have

$$L[f; P] \leq U[f; P]. \qquad (1)$$

Note. Geometrically, $U[f; P]$ is the sum of the areas of circumscribed rectangles and $L(f; P)$ is the sum of the areas of inscribed rectangles of the curve $y = f(x)$ corresponding to the partition P, if f is a continuous and non-negative function on $[a, b]$.



$L[f; P]$



$U[f; P]$

From the definition of partition, we have the following property.

9.3 DAURBBOUX'S THEOREM

Theorem 1. For any partition P of $[a, b]$, we have

$$m[f; P](b-a) \leq L[f; P] \leq U[f; P] \leq M[f; P](b-a).$$

Proof. For each $k = 1, 2, 3, \dots, n$, we have

$m[f;\ I]\ |I_k| \leq m[f;\ I_k]\ |I_k| \leq M[f;\ I_k]\ |I_k| \leq M[f;\ I]\ |I_k|$.

Hence, $[a, b]$ for the partition P, we get

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$m[f;\ I]\ \sum_{k=1}^{n}|I_k| \leq \sum_{k=1}^{n} m[f;\ I_k]\ |I_k| \leq \sum_{k=1}^{n} M[f;\ I_k]\ |I_k| \leq M[f;\ I]\ \sum_{k=1}^{n}|I_k|$.

From this we get

$$m[f;I](b-a) \leq L[f;P] \leq U[f;P] \leq M[f;I](b-a).$$

From the above inequalities, we conclude that the set of all lower sums $L[f;P]$ is bounded above for every P and an upper bound for the lower sums is the real number $M[f;P](b-a)$. Similarly, $U[f;P]$ is bounded below for every P and a lower bound is the real number $m[f;P](b-a)$.

Definition 4. Let f be a bounded function on the closed and bounded interval $[a,b]$. The upper integral of f over $[a,b]$ is defined as

$$\int_{a}^{-b} f(x)dx = \text{glb}_P\, U[f;P] \qquad (2)$$

where glb is taken over all possible partitions P of $[a,b]$. Similarly the lower integral of f over $[a,b]$ is defined as

$$\int_{-a}^{b} f(x)dx = \text{lub}_P\, L[f;P] \qquad (3)$$

where lub is taken over all partitions P of $[a,b]$.

For simplicity the upper and lower integrals of f in $[a,b]$ are denoted by $\int_{a}^{-b} f$ and $\int_{-a}^{b} f$.

Since the set of all lower sums $L[f;P]$ for all possible partitions is bounded above by Theorem 1, the lower integral exists. Similarly, the set of all upper sums $U[f;P]$ is bounded below for every partition, the upper integral exists. Further from the inequality (1), we have $\int_{-a}^{b} f < \int_{a}^{-b} f.$ (4)

Definition 5. If f is a bounded function on the closed and bounded interval $[a,b]$, f is said to be Riemann integrable on $[a,b]$ provided

$$\int_{-a}^{b} f = \int_{a}^{-b} f.$$

The common value of the upper and lower integrals is denoted by $\int_{a}^{b} f$ or $\int_{a}^{b} f(x)dx$ and called the Riemann integral of f with respect to x in $[a,b]$.

Example 1. Each constant function $f(x) = c$ is Riemann integrable on any interval $[a,b]$.

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Let $P: a = x_0 < x_1 < x_2 \ldots < x_{k-1} < x_k \ldots < x_{n-1} < x_n = b$ be a partition of $[a, b]$. Then

$$U[f;P] = \sum M[f;I_k]\,|I_k| = \sum c\,|I_k| = c(b-a)$$

$$L[f;P] = \sum m[f;I_k]\,|I_k| = \sum c\,|I_k| = c(b-a)$$

Since P is arbitrary, it follows that $U[f;P] = L[f;P] = c(b-a)$ for every partition P of $[a,b]$ and $\int_a^{-b} f = c(b-a)$ and $\int_{-a}^{b} f = c(b-a)$ are equal.

Thus f is Riemann integrable and $\int_a^b f = c(b-a)$.

Example 2. Let $f(x) = x$ $(0 \leq x \leq 1)$. Let σ be the partition $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ of $[0,1]$. Compute $U[f;\sigma]$ and $L[f;\sigma]$.

For the given partition σ, the component intervals of $[0,1]$ are

$$I_1 = \left[0, \frac{1}{4}\right], \quad I_2 = \left[\frac{1}{4}, \frac{2}{4}\right], \quad I_3 = \left[\frac{2}{4}, \frac{3}{4}\right], \quad I_4 = \left[\frac{3}{4}, 1\right]$$

$$M[f;I_1] = \frac{1}{4}, \quad M[f;I_2] = \frac{2}{4}, \quad M[f;I_3] = \frac{3}{4}, \quad M[f;I_4] = 1$$

$$m[f;I_1] = 0, \quad m[f;I_2] = \frac{1}{4}, \quad m[f;I_3] = \frac{2}{4}, \quad m[f;I_4] = \frac{3}{4}$$

Hence, let us find $U[f,\sigma]$ and $L[f,\sigma]$.

Now, $U[f;\sigma] = \sum_{k=1}^{4} M[f;\ I_k]\ |I_k|$

$$= \frac{1}{4}.\frac{1}{4} + \frac{2}{4}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4} + 1.\frac{1}{4}$$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$$

$$L[f;\sigma] = \sum_{k=1}^{4} m[f;\ I_k]\ |I_k$$

$$= \frac{1}{4}.0 + \frac{1}{4}.\frac{1}{4} + \frac{2}{4}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4}$$

$$= 0 + \frac{1}{16} + \frac{2}{16} + \frac{3}{16} = \frac{6}{16} = \frac{3}{8}$$

Hence, we get $[f;\sigma] = \frac{5}{8}$, $L[f;\sigma] = \frac{3}{8}$.

NOTES

Example 3. Let $f(x) = x^2$. For each $n \in N$, let σ_n be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$ of $[0,1]$. Compute $\lim_{n\to\infty} U[f;\sigma_n]$ and $\lim_{n\to\infty} L[f;\sigma_n]$.

Now the component intervals of the partition P are,

$$I_1 = \left[0, \frac{1}{n}\right], \quad I_2 = \left[\frac{1}{n}, \frac{2}{n}\right], \quad I_3 = \left[\frac{2}{n}, \frac{3}{n}\right], \dots \ , I_4 = \left[\frac{n-1}{n}, \frac{n}{n}\right]$$

$$M[f;I_1] = \frac{1}{n^2}, \quad M[f;I_2] = \left(\frac{2}{n}\right)^2, \quad M[f;I_3] = \left(\frac{3}{n}\right)^2$$

$$M[f;I_4] = \left(\frac{4}{n}\right)^2, \dots, \quad M[f;I_n] = \left(\frac{n}{n}\right)^2$$

$$m[f;I_1] = 0, \quad m[f;I_2] = \left(\frac{1}{n}\right)^2, \quad m[f;I_3] = \left(\frac{2}{n}\right)^2$$

$$m[f;I_4] = \left(\frac{3}{n}\right)^2, \dots, \quad m[f;I_n] = \left(\frac{n-1}{n}\right)^2$$

Hence, $$U[f;\sigma_n] = \frac{1}{n^2}.\frac{1}{n} + \left(\frac{2}{n}\right)^2.\frac{1}{n} + \left(\frac{3}{n}\right)^2.\frac{1}{n} + \dots + \left(\frac{n}{n}\right)^2.\frac{1}{n}$$

$$= \frac{1}{n^3}[1 + 2^2 + 3^2 + \dots + n^2].$$

$$U[f;\sigma_n] = \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \text{ and hence, } \lim_{n\to\infty} U[f;\sigma_n] = \frac{1}{3}.$$

$$L[f;\sigma_n] = 0.\frac{1}{n} + \left(\frac{1}{n}\right)^2.\frac{1}{n} + \left(\frac{2}{n}\right)^2.\frac{1}{n} + \left(\frac{3}{n}\right)^2.\frac{1}{n} + \dots + \left(\frac{n-1}{n}\right)^2.\frac{1}{n}$$

$$= \frac{1}{n^3}[1 + 2^2 + 3^2 + \dots + (n-1)^2]$$

$$= \frac{1}{n^3}\left[\frac{n(n-1)(2n-1)}{6}\right] \text{ and hence , } \lim_{n\to\infty} L[f;\sigma_n] = \frac{1}{3}.$$

The partition P^* of $[a,b]$ is called a refinement of P, if each point of subdivision x_i of P is also a point of subdivision of P^* . The partition P^* is called the common refinement of the partitions P_1 and P_2, if P^* is the refinement of both P_1 and P_2. Every pair of partitions P_1 and P_2 has common refinement. For example $P^* = P_1 \cup P_2$ consisting of the points of P_1 and P_2 is a common refinement of both P_1 and P_2. The length of the largest of the component intervals of the partition is denoted by $||P||$. That is $||P|| = \max_{1\le k\le n}(x_k - x_{k-1})$. Using these we have the following theorem.

Theorem 2. Let f be bounded function on $[a, b]$. Then every upper sum for f is greater than or equal to every lower sum for f. That is, if P_1 and P_2 are any two partitions of $[a, b]$, then $U[f; P_1] \geq L[f;\ P_2]$.

Proof. To prove this, first we shall establish that if ${P_1}^*$ is any refinement of P_1 and ${P_2}^*$ is a refinement of P_2, then

$$U[f; P_1] \geq U[f; {P_1}^*\,] \quad \text{and} \quad L[f; P_2] \geq L[f; {P_2}^*\,]. \qquad (1)$$

That is, any refinement of the given partition decreases the upper sum and increases the lower sum. It is enough if we prove the case where ${P_1}^*$ is obtained from P_1 by adding only one point of subdivisions. We suppose that P has component intervals $I_1, I_2, \dots I_k, \dots, I_n$ and ${P_1}^*$ has component intervals. $I_1, I_2, \dots {I_k}^*, {I_k}^{**}, \dots, I_n$ where $I_k = {I_k}^* \cup {I_k}^{**}$ and $|I_k| = |{I_k}^*| + |{I_k}^{**}|$. since ${I_k}^* \subset I_k$, we have $M[f; {I_k}^*] \leq M[f; I_k]$ and $M[f; {I_k}^{**}] \leq M[f; I_k]$.

Hence we have,

$$U[f; {P_1}^*] = \sum_{j=1, j\neq k}^{n} M[f; I_j].\,|I_j| + M[f; {I_k}^*].\,|{I_k}^*| + M[f; {I_k}^{**}].\,|{I_k}^{**}|$$

$$\leq \sum_{j=1, j\neq k}^{n} M[f; I_j].\,|I_j| + M[f; I_k].\,(|{I_k}^*| + |{I_k}^{**}|)$$

$$\leq U[f; P_1].$$

Hence, we have $U[f; {P_1}^*] \leq U[f;\ P_1]$.

In a similar manner , we can show that $L[f; P_2] \leq L[f;\ {P_2}^*]$.

Now since $P_1 \cup P_2$ is a refinement of both P_1 and P_2, we have from the above

$$U[f; P_1] \geq\ U[f; P_1 \cup P_2] \geq L[f; P_1 \cup P_2] \geq L[f; P_2].$$

Note. From the theorem, we get $glb\, U[f; P_1] \geq lub\, L[f; P_1]$ where glb and lub are both taken over all partitions of $[a, b]$.

For, if P_2 is any partition of $[a, b]$, then from the theorem, $L[f; P_1]$ is the lower bound for the set of all upper sums $U[f; P_1]$. Hence we get $L[f; P_2] < glb\, U[f; P_1]$ for every partition P_2. But $b\, U[f; P_1]$, is the upper bound for the set of all lower sums $L[f; P_2]$. Hence, we get

NOTES

$$lub\ L[f;P_2] \leq glb\ U[f;P_1].$$

Using the above inequality, we get immediately from the definition of upper and lower integrals

$$\int_{-a}^{b} f(x)dx \leq \int_{a}^{-b} f(x)dx.$$

9.4 CONDITIONS FOR INTEGRABILITY

Theorem 3. Let f be a bounded function on the closed bounded interval $[a, b]$. Then f is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a subdivision P of $[a, b]$ such that

$$U[f;P] - L[f;P] < \varepsilon.$$

Proof. First suppose that for the given $\varepsilon > 0$, there exists a partition P such that (1) is true. Then since

$$\int_{a}^{-b} f(x)dx \leq U[f;P] \text{ and } \int_{-a}^{b} f(x)dx \geq L[f;P].$$

Hence, using these two in (1), we get $\int_{a}^{-b} f(x)dx - \int_{-a}^{b} f(x)dx < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we get from the above

$$\int_{a}^{-b} f(x)dx \leq \int_{-a}^{b} f(x)dx$$

From the previous note, we get $\int_{-a}^{b} f(x)dx \leq \int_{a}^{-b} f(x)dx$.

Hence we have from (2) and (3), $\int_{-a}^{b} f(x)dx = \int_{a}^{-b} f(x)dx$, so that f is Riemann integrable in $[a, b]$.

Conversely, suppose f is Riemann integrable in $[a, b]$.

Then $\int_{a}^{-b} f = \text{glb}_P\, U[f;P_1] = \text{lub}_P\, L[f;P_2] = \int_{-a}^{b} f$

Given $\varepsilon > 0$, from the definition of glb, we can choose a partition P_1 such that

$$U[f;P_1] < \int_{a}^{-b} f + \frac{\varepsilon}{2}.$$

In the same manner, we can choose a partition P_2 such that

$$L[f;P_2] > \int_{-a}^{b} f - \frac{\varepsilon}{2}.$$

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Using the fact that f is Riemann integrable, we get

$$L[f;P_2] + \frac{\varepsilon}{2} > U[f;P_1] - \frac{\varepsilon}{2}.$$

Now, considering the common partition of P_1 and P_2

$$L[f;P_1 \cup P_2] + \frac{\varepsilon}{2} > U[f;P_1 \cup P_2] - \frac{\varepsilon}{2}.$$

Now, considering $P_1 \cup P_2$ as single partition P, we get

$$U[f;P] - L[f;P] < \varepsilon.$$

This completes the proof of the theorem.

The following theorems illustrate the use of the above criterion of integrability for a bounded function in a closed and bounded interval.

9.5 INTEGRABILITY OF CONTINUOUS & MONOTONIC FUNCTIONS

Theorem 4. Every continuous function on $[a,b]$ is Riemann integrable.

Proof. Suppose f is continuous on $[a,b]$ and let $\varepsilon > 0$ be given. We shall show that corresponding to this $\varepsilon > 0$, there exists a partition P for $[a,b]$ such that

$$U[f;P] - L[f;P] < \varepsilon.$$

By the uniform continuity of f on $[a,b]$, there is a $\delta > 0$ such that

$|f(x) - f(y)| < \frac{\varepsilon}{b-a}$, whenever $x, y \in [a,b]$ with $|x - y| < \delta$.

Let P be any partition of $[a,b]$ with $||P|| < \delta$. By the property of continuous function on the closed interval $[x_{k-1}, x_k]$, there exists points x_k' and $x_k'' \in [x_{k-1}, x_k]$ such that

$$f(x_k') = M_k \text{ and } (x_k'') = m_k.$$

Now, $|x_k' - x_k''| < |x_k, x_{k-1}| = |I_k| < ||P|| < \delta$.

Hence, $M_k - m_k = |f(x_k') - f(x_k'')| < \frac{\varepsilon}{b-a}$ for $k = 1,2,\dots,n$.

Hence, $U[f;P] - L[f;P] = \sum_{k=1}^{n} \big[M[f;I_k] - m[f;I_k]\big] |I_k|$

$$= \sum_{k=1}^{n} |f(x_k') - f(x_k'')|\, |I_k|$$

NOTES

$$\leq \frac{\varepsilon}{b-a}\sum_{k=1}^{n} |I_k| = \frac{\varepsilon}{b-a}(b-a) = \varepsilon\,.$$

Hence, f is Riemann integrable on $[a,b]$.

Theorem 5. If f is monotone on $[a,b]$, then it is Riemann- integrable on $[a,b]$.

Proof. If f is constant on $[a,b]$, then it is Riemann integrable on $[a,b]$ by Example 1.

Since we can give a similar proof for monotonic decreasing case, we assume that f is monotonic increasing on $[a,b]$ and $f(a) < f(b)$. Let $\varepsilon > 0$ be given. We shall show that there exists a partition P on $[a,b]$ for which $U[f;P] - L[f;P] < \varepsilon$. Let P be any partition on $[a,b]$ with $||P|| < \frac{\varepsilon}{f(b)-f(a)}$. Then since f is increasing on $[a,b]$, we have

$$M[f;I_k] = f(x_k) \text{ and } m[f;I_k] = f(x_{k-1}) \text{ for } k = 1, 2, \dots, n.$$

Hence,
$$U[f;P] - L[f;P] = \sum_{k=1}^{n} M[f;I_k]|I_k| - \sum_{k=1}^{n} m[f;I_k]|I_k|$$

$$= \sum_{k=1}^{n}\big[M[f;I_k] - m[f;I_k]\big]|I_k|$$

$$= \sum_{k=1}^{n}[f(x_k) - f(x_{k-1})]|I_k|$$

$$< \frac{\varepsilon}{f(b)-f(a)}\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})]$$

$$= \frac{\varepsilon}{f(b)-f(a)}[f(x_n) - f(x_0)] = \varepsilon.$$

Hence, by Theorem 3, f is Riemann integrable on $[a,b]$.

Check your progress

1) Define Riemann integrable
2) Provide the conditions for integrability
3) What is the integrability of continuous function?
4) What is the integrability of the monotone function?

9.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1) If f is a bounded function on the closed and bounded interval $[a,b]$, f is said to be Riemann integrable on $[a,b]$ provided

$$\int_{-a}^{b} f = \int_{a}^{-b} f.$$

NOTES

The common value of the upper and lower integrals is denoted by $\int_a^b f$ or $\int_a^b f(x)dx$ and called the **Riemann integral** of f with respect to x in $[a,b]$.

2) Let f be a bounded function on the closed bounded interval $[a,b]$. Then f is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a subdivision P of $[a,b]$ such that

$$U[f;P] - L[f;P] < \varepsilon.$$

3) Every continuous function on $[a,b]$ is Riemann integrable.

4) If f is monotone on $[a,b]$, then it is Riemann- integrable on $[a,b]$.

9.7 SUMMARY

- Let I be a bounded and closed interval of . Let f be a bounded real valued function defined on I. Let us define the following
  $$\boldsymbol{M[f;I] = \text{lub}_{x\in I} f(x),\ \ m[f;I] = \text{glb}_{x\in I} f(x).}$$
- A **partition** P of $[a,b]$ is a finite subset$\{x_0, x_1, x_2, \dots, x_n\}$ of $[a,b]$ such that P: $a = x_0 < x_1 < x_2 < \dots < x_n = b$.
- The points $x_0, x_1, x_2, \dots, x_n$ are called the points of **sub-division of $[a,b]$**. The closed interval
  $$I_1 = [x_0, x_1], \quad I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$
- are called the **component** intervals of $[a,b]$. For the partition P, we have in the above notation
  $$M[f;\ I] = \underset{x\in I}{\text{lub}}\, f(x),\ \ m[f;I] = \underset{x\in I}{\text{glb}}\, f(x)$$

  where $k = 1, 2, 3, \dots, n$.
- From the definition of partition we have
  $$\boldsymbol{m[f;\ I] \le m[f;\ I_k] \le M[f;\ I_k] \le M[f;\ I]} \text{ **for each** } \boldsymbol{k}.$$
- Let f be a bounded function on the closed bounded interval $[a,b]$ and let P be any partition of $[a,b]$. We define the **upper sum** of f corresponding to the partition P as
  $$U[f;P] = \sum_{k=1}^{n} M[f;\ I_k]\ |I_k|.$$
- Similarly, the **lower sum** of f is defined as
  $$L[f;P] = \sum_{k=1}^{n} m[f;\ I_k]\ |I_k|.$$
- $L[f;P] \le U[f;P]$.
- Let f be a bounded function on the closed and bounded interval $[a,b]$. **The upper integral** of f over $[a,b]$ is defined as
  $$\int_a^{-b} f(x)dx = \text{glb}_P\, U[f;P] \qquad (2)$$
  where glb is taken over all possible partitions P of $[a,b]$. Similarly the **lower integral** of f over $[a,b]$ is defined as
  $$\int_{-a}^{b} f(x)dx = \text{lub}_P\, L[f;P] \qquad (3)$$
  where lub is taken over all partitions P of $[a,b]$.
  - If f is a bounded function on the closed and bounded interval $[a,b]$, f is said to be Riemann integrable on $[a,b]$ provided
    $$\int_{-a}^{b} f = \int_a^{-b} f.$$

NOTES

The common value of the upper and lower integrals is denoted by $\int_a^b f$ or $\int_a^b f(x)dx$ and called the **Riemann integral** of f with respect to x in $[a,b]$.

- Let f be a bounded function on the closed bounded interval $[a,b]$. Then f is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a subdivision P of $[a,b]$ such that

$$U[f;P] - L[f;P] < \varepsilon.$$

- Every continuous function on $[a,b]$ is Riemann integrable.
- If f is monotone on $[a,b]$, then it is Riemann- integrable on $[a,b]$.

9.8 KEYWORDS

- If f is a bounded function on the closed and bounded interval $[a,b]$, f is said to be Riemann integrable on $[a,b]$ provided

$$\int_{-a}^{b} f = \int_{a}^{-b} f.$$

The common value of the upper and lower integrals is denoted by $\int_a^b f$ or $\int_a^b f(x)dx$ and called the **Riemann integral** of f with respect to x in $[a,b]$.

9.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

1)Find whether f is Riemann integrable on [0,1] and justify your answers.

i) $f(x) = \frac{1}{x+3}$

ii) $f(x) = |x - \frac{1}{2}|$

iii) $f(x) = [x]$

2) If f is continuous on [0,1], prove that $\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f$.

9.10 FURTHER READINGS

1) Arumugam & Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2010.

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UNIT 10 INTEGRAL FUNCTIONS

Structure

NOTES

10.0 INTRODUCTION

All the different conditions we have stated in the previous discussion for Riemann integrability of bounded functions on a bounded and closed interval $[a, b]$ are only sufficient. In this section, we shall first explain how the concept of continuity is related to the Reimann-integrability and characteristic the Reimann integrable functions by using functions continuous almost everywhere. For such a characterization, we shall first introduce the concept of a set of measure zero. If I is an interval of real numbers, let $|I|$ denote the length of the interval.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss existence of Riemann integral
- Discuss properties of the Riemann Integral
- Discuss the Continuity & Derivability of integral functions

10.2 EXISTENCE OF RIEMANN INTEGRAL

Defenition 1. A subset E of R is said to be of measure zero if for each $\varepsilon > 0$, there exists a finite or countable number of open intervals (I_n) such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_n |I_n| < \varepsilon$.

NOTES

Note. Hence E is a set of measure zero, if given $\varepsilon > 0$, E can be covered by a union of open intervals whose total length is less than ε. It is easy to see that a set consisting of one point of measure zero.

The following theorems give some properties of sets of measure zero which we need in our discussion.

Theorem 1. If each of $E_1, E_2, \ldots,$ of R is of measure zero, then $\bigcup_{n=1}^{\infty} E_n$ is also of measure zero.

Proof. Let us fix $\varepsilon > 0$. Since E_n is of measure zero, for each positive integer n, there exists a finite or a countable number of open intervals which cover E_n and whose total length is less than $\frac{\varepsilon}{2^n}$. Then the union of all such open intervals for all n covers $\bigcup_{n=1}^{\infty} E_n$and the length of all these countably many intervals is less than

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^n} + \cdots < \varepsilon.$$

Hence, we get $\bigcup_{n=1}^{\infty} E_n$ is a set of measure zero.

Corollary. Every countable set of R is a set of measure zero.

This follows by using the fact that one point sets are of measure zero in the theorem

Definition 2. If a property is true on $[a, b]$ expect on a set of measure zero, then the property is said to be true almost everywhere on $[a, b]$ or for lmost all points of $[a, b]$. That is, the set of points of $[a, b]$ at which the property is not true is a set of measure zero.

Thus if f is continuous almost everywhere in $[a, b]$, then the set of points E of $[a, b]$ at which f is not continuous is a set of measure zero.

Example1. If A is not of measure zero, if $B \subset A$, and if B is of measure zero, prove that $A - B$ is not of measure zero.

If $A - B$ is not of measure zero, then let it be of measure zero. Now $A = (A - B) \cup B$. By hypothesis B is of measure zero and by assumption $(A - B)$ is of measure zero. Since union of a finite number of sets of measure zero is of measure zero. $A = (A - B) \cup B$ is a set of measure zero contradicting that A is not a set of measure zero. Hence $A - B$ is not a set of measure zero.

Example 2. If $a < b$, prove that $[a, b]$ cannot be covered by a finite number of open intervals whose total length is less than $b - a$.

NOTES

Since $[a, b]$ is a bounded and closed interval, every open covering of $[a, b]$ contains a finite subcovering. Hence, it is enough if we prove that for a finite collection of open intervals covering $[a, b]$, $\sum_{n=1}^{k}|I_n| > b - a$.

Since a is contained in $\bigcup_{n=1}^{k} I_n$, there must be one of the I_n's which contains a. Let this be the interval (a_1, b_1). We have $a_1 < a < b_1$. If $b_1 < b$, then $b_1 \in [a, b]$ and since $b_1 \in [a, b]$, there must be an interval (a_2, b_2) in the collection (I_n) such that $b \in (a_2, b_2)$. That is $a_2 < b_1 < b_2$. Proceeding in this manner, we get a sequence $(a_1, b_1), \ldots, (a_k, b_k) \ldots$ from the collection (I_n) such that $a_i < b_{i-1} < b_i$.

Since (I_n) is a finite collection, the above process must terminate with some interval (a_k, b_k). But it ends only when $b \in (a_k, b_k)$. That is $a_k < a < b_k$. Thus

$$\sum_{n=1}^{k} |I_n| > \sum_{n=1}^{k} |(a_n, b_n)|$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1)$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots - (a_2 - b_1) - a_1 > b_k - a_1.$$

Since $a_i < b_{i-1}$. But $b_k > b$ and $a_1 < a$ and so $b_k - a_1 > b - a$, from which we have $\sum_{n=1}^{k} |I_n| > (b - a)$.

Example 3. If $a < b$, prove that (a, b) is not of measure zero.

Now from Example 2, $[a, b]$ is not of measure zero.

But $\{a, b\} \in [a, b]$ is of measure zero. By Example 1, $[a, b] - \{a, b\} = (a, b)$ is not of measure zero.

Example 4. Prove the following:

i. The set of rational numbers Q is of measure 0.
ii. The set of all irrational numbers is not of measure zero.

Since Q is a countable set, the set Q of all rational number is of measure zero follows by Corollary 1 of Theorem 1.

The set R of all real numbers is not of measure zero. $Q \subset R$ is of measure zero. Hence, $Q^c = R - Q$ is not of measure zero by Example 1.

NOTES

Theorem 2. Let f be a bounded function on the closed and bounded interval $[a, b]$. Then f is Riemann integrable if and only if f is continuous at almost every point in $[a, b]$.

Proof. Let us first suppose that f is Riemann integrable in $[a, b]$. Then we have to show that the set D of points in $[a, b]$ at which f is not continuous is of measure zero. Now by Theorem 1 of 5.9, $x \in D$ if and only if $\omega[f; x] > 0$. Hence let $D = \cup_{n=1}^{\infty} D_m$ where each D_m is the set of all points x in $[a, b]$ such that $\omega[f; x] > \frac{1}{m}$. To prove that D is of measure zero, it is enough if we show that each D_m is of measure zero.

Let m be fixed. Since f is Riemann integrable, given $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U[f; P] - L[f; P] < \frac{\varepsilon}{2m}.$$

Hence, if $I_1, I_2, \dots, I_n$ are the closed component interval of P, we have

$$\sum_{k=1}^{n} \omega[f; I_k]\,|I_k| = \sum_{k=1}^{n} M[f; I_k]\,|I_k| - \sum_{k=1}^{m} m[f; I_k]\,|I_k|$$

$$= U[f; P] - L[f; P] < \frac{\varepsilon}{2m} \text{ by Hypothesis.}$$

Hence, we have from the above

$$\sum_{k=1}^{n} \omega[f; I_k]\,|I_k| < \frac{\varepsilon}{2m} \quad \dots\dots\dots\dots\dots(1)$$

Now let $D_m = D'_m \cup D''_m$ where D'_m is the set of points of D_m that are the points of the partition and $D''_m = D - D'_m$. Since there are only finite number of point of the partition in D'_m, we see that $D'_m \subset J_1 \cup J_2 \cup \dots \cup J_p$, where J'_i s are the open subintervals such that

$$|J_1| + |J_2| + \cdots + |J_p| < \frac{\varepsilon}{2}$$

But if $x \in D''_m$, then x is an interior point of some I_k. Hence, we have

$$\omega[f; I_k] \geq \omega[f; x] > \frac{1}{m}.$$

If $I_{k_1}, I_{k_2}, \dots, I_{k_r}$ are those component intervals of P which contain a point of D''_m in their interior, we have

$\frac{1}{m}(|I_{k_1}| + |I_{k_2}| + \cdots + |I_{k_r}|) < \omega[f; I_{k_1}]|I_{k_1}| + \cdots + \omega[f; I_{k_r}]|I_{k_r}|$.

Hence, we have from (1),

$$(|I_{k_1}| + |I_{k_2}| + \cdots + |I_{k_r}|) < \frac{\varepsilon}{2}.$$

Since D''_m is covered by the interiors of $I_{k_1}, I_{k_2}, \dots, I_{k_r}$ and since D'_m is covered by $J_1, J_2, \dots, J_p$, it follows that D_m is covered by a finite number of intervals, sum of whose lengths is less than ε.

Hence, $D_m = D'_m \cup D''_m$ is a set of measure zero.

To prove the converse, we need the following lemma.

Lemma. If $\omega[f; x] < a$ for each x in a closed bounded interval J, then there σ of J such that

$$U[f; \sigma] - L[f; \sigma] < a|J| \quad \text{....................(2)}$$

Proof. For each $x \in J$, there is an open interval I_x containing x such that $\omega[f; x] < a$. Since J is compact, a finite number of these I_x will cover J. Let σ be the set of end points of these I_x. If $I_1, I_2, \dots, I_n$ are the component intervals of σ, we have $\omega[f; x] < a$ for $k = 1,2, \dots, n$ and hence (2) follows easily.

Now let us assume that f is continuous at almost every point of $[a, b]$, we have to show that $f$is Riemann integrable on $[a, b]$. Given $\varepsilon > 0$, choose aa positive integer m such that $\frac{b-1}{m} < \frac{\varepsilon}{2}$.

If D_m is defined as the first part of the proof D_m is of measure zero. Hence $D_m \subset \bigcup_{n=1}^{\infty} I_n$ where each I_n is an open subinterval of $[a, b]$. Since $\omega[f; [a, b]] > 0$, let us take

$$\sum_{n=1}^{\infty} |I_n| < \frac{\varepsilon}{2\omega[f; [a, b]]}.$$

But we know that D_m is closed in R. Hence D_m is a closed subset of $[a, b]$ and is thus compact. Therefore, a finite number of intervals of the (I_n) will cover D_m. Let them be $I_{n_1}, I_{n_2}, \dots, I_{n_k}$.

Now $[a, b] - (I_{n_1} \cup I_{n_2} \cup \dots \cup I_{n_k})$ is a union of closed intervals $J_1, J_2, \dots, J_p$.

That is, $[a, b] = (I_{n_1} \cup I_{n_2} \cup \dots \cup I_{n_k} \cup J_1 \cup J_2 \dots J_p)$

NOTES

Since no interval $J_i (i = 1, 2, \dots, p)$ contains a point of D_m, there exists (by the lemma) a subdivision σ_i of J_i such that

$$U[f; \sigma_i] - L[f; \sigma_i] < \frac{|J_i|}{m}.$$

Now define a partition P of $[a, b]$ as $P = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_p$. Then the component intervals of P are the component intervals of $\sigma_1, \sigma_2, \dots, \sigma_p$ together with $I_{n_1}, I_{n_2}, \dots, I_{n_k}$. Hence we have $U[f; \sigma] - L[f; \sigma] = \sum_{i=1}^{p} U[f; \sigma_i] - L[f; \sigma_i] + \sum_{i=1}^{k} \left[M[f; I_{n_i}] - m[f; I_{n_i}]\right] |I_{n_i}|$

$$\leq \frac{1}{m} \sum_{i=1}^{p} |J_i| + \sum_{i=1}^{k} \omega[f; I_{n_i}]|I_{n_i}|$$

$$\leq \frac{b-a}{m} + \omega[f; [a, b]] \sum_{i=1}^{k} |I_{n_i}|$$

$$< \frac{\varepsilon}{2} + \omega[f; [a, b]] \frac{\varepsilon}{2\omega[f; [a, b]]} = \varepsilon.$$

Hence, by Theorem 3 of 8.1., f is Riemann integrable on $[a, b]$. So the proof of the theorem is complete.

Example 5. Determine whether the following functions are Riemann integrable

i. $f(x) = \sin\frac{1}{x}$ for $0 < x \leq 1$ and $f(0) = 2$

ii. $f(x) = n$, if $x = \frac{1}{n}$ when $n = 1,2,3 \dots,$ and $f(x) = 0$ otherwise for $x \in [0,1]$.

iii. $f(x) = x^2$ if x is rational and $f(x) = 0$ if x is irrational

iv. Let $f(x) = 0$ for x in $[0,1]$ and $f(x) = 1$ for $x \in \left\{0, \frac{1}{10}, \frac{2}{10}, \dots, \right\}$.

(i) 0 is the only point of discontinuity of f in [0,1]. Hence, f is continuous almost everywhere in [0,1]. Son it is Riemann integrable in [0,1].

(ii) The function is continuous in [0,1] except at the points $x = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. Since the point of discontinuous are countable and any countable set is a set of measure zero, the function is continuous almost everywhere on [0,1]. Therefore, f is Riemann integrable.

(iii) Every point of $[0,1]$ is a point of discontinuity of f. Hence, f is discontinuous throughout the interval $[0,1]$. So f is not Riemann integrable on $[0,1]$.

(iv) From the definition, f is continuous everywhere except at the points $\left\{0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots\right\}$ which is countable and so that function is continuous almost everywhere. Hence, f is Riemann integrable on $[0,1]$.

Example 6. Find $\omega[f;0]$ for the function f defined on $[0,1]$ as follows:

$$f(x) = \frac{\sin x}{x} \text{ if } x \neq 0 \text{ and } f(x) = 1 \text{ if } x = 0.$$

In the neighbourhood of $x = 0$, $f(x) = 0$ and by hypothesis $f(0) = 1$. Hence $\omega[f;0] = 0$.

Example 7. Let C be an arbitrary countable infinite subset of $[0,1]$. Find a function f defined and bounded on $[0,1]$ such that $D(f) = C$. Is f Riemann integrable on $[0,1]$?

Let $C = \{x_1, x_2, \ldots\} \subset [0,1]$.

Let us define the function f as follows:

$$f(x) = \frac{1}{n} \text{ if } x = x_n \text{ and } f(x) = 0 \text{ otherwise.}$$

The function is continuous in $[0,1]$ except at the set of points $\{x_1, x_2, x_3, \ldots\}$ which is a set of measure zero. Hence, by the Theorem 2, f is integrable on $[0,1]$.

10.2 PROPERTIES OF THE RIEMANN INTEGRAL

In this section, we shall consider the Riemann integrable functions on a bounded closed interval $[a,b]$ and establish some of their properties. We shall denote the set of all Riemann integrable functions on $[a,b]$ by $R[a,b]$.

Theorem 1. If f is Riemann integrable on $[a,b]$ and c is any real number, then cf is Riemann integrable and

$$\int_a^b cf = c\int_a^b f.$$

NOTES

Proof. If $c = 0$, the theorem is obvious. Since cf is continuous almost everywhere on $[a, b]$, cf is Riemann integrable on $[a, b]$. If I is any subinterval of $[a, b]$ and $c > 0$,

$$M[cf, I] = cM[f; I].$$

Hence, for any partition P of $[a, b]$ we get

$$U[cf; P] = cU[f; P].$$

Therefore, taking glb on both sides of the above,

$$\int_a^b cf = c \int_a^b f, \text{ where } c > 0. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Hence, we have proved the theorem when $c > 0$.

For any interval I, we have $M[-f; I] = -m[f; I]$. Hence,

$$\int_a^b (-f) = glb\ U[-f, P] = glb - L[f, P]$$

$$= -lub\ L[f, P] = -\int_a^b f.$$

From this, we have

$$\int_a^b (-f) = -\int_a^b f. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If $d < 0$, then $c = -d > 0$ and so by (2) and (1)

$$\int_a^b df = \int_a^b -(cf) = -\int_a^b cf = -c\int_a^b f = d\int_a^b f.$$

This completes the proof of the theorem.

Theorem 2. If $f \in R[a, b]$ and $g \in R[a, b]$, then $f + g \in R[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

NOTES

Proof. By Theorem, the sets $D(f)$ and $D(g)$ of points of discontinuities of f and g are both of measure zero. By Theorem, the set $D(f) \cup D(g)$ is of measure zero.

If $x \in [a,b] - [D(f) \cup D(g)]$, then f, g and hence $f + g$ are continuous at x. Thus $f + g$ is continuous at almost every point in $[a,b]$ and so $f + g \in R[a,b]$.

If J is any interval contained in $[a,b]$ and if $y \in J$, we have $f(y) + g(y) \le M[f;J] + M[g;J]$.

For any partition P of $[a,b]$, we have by using the above results

$$\int_a^b (f+g) \le U[f+g;P]$$

$$\le U[f;P] + U[g;P] \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But given $\varepsilon > 0$, there is a partition P_1 of $[a,b]$ such that

$$U[f;P_1] < L[f;P_1] + \frac{\varepsilon}{2} < \int_a^b f + \frac{\varepsilon}{2}.$$

Also there is a subdivision P_2 of $[a,b]$ such that

$$U[g;P_2] < L[g;P_2] + \frac{\varepsilon}{2} < \int_a^b g + \frac{\varepsilon}{2}.$$

If $P = P_1 \cup P_2$, then P is a refinement of P_1 and P_2. So we get

$$U[f;P] < \int_a^b f + \frac{\varepsilon}{2}, U[g;P] < \int_a^b g + \frac{\varepsilon}{2}.$$

From (1), we get $\int_a^b (f+g) < \int_a^b f + \int_a^b g + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, this proves that

NOTES

$$\int_a^b (f+g) < \int_a^b f + \int_a^b g \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Since f and g were any Riemann integrable functions, we can substitute $-f, -g$ for f and g in (2).

Hence, $\int_a^b (-f-g) < \int_a^b (-f) + \int_a^b (-g)$.

By using the above theorem we have

$$-\int_a^b (f+g) < -\left(\int_a^b f + \int_a^b g\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now multiply both sides of (3) by -1. This reserves the inequality and so

$$\int_a^b (f+g) \geq \int_a^b f + \int_a^b g \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Hence, the theorem follows from (2) and (4).

Theorem 3. If $f \in R[a,b]$ and if $f(x) \geq 0$ almost everywhere on $[a,b]$, then

$$\int_a^b f \geq 0.$$

We have

$$m[f;I](b-a) \leq \int_a^b f \leq M[f;I](b-a).$$

If $f(x) > 0$ for every $x \in [a,b]$, then $m[f;I] > 0$. Using this in the above inequality, we get

$$\int_a^b f \geq 0.$$

From the above theorem, we have the following corollaries.

NOTES

Corollary 1. $f \in R[a,b]$ and $g \in R[a,b]$ and if $f(x) \leq g(x)$ almost everywhere on $[a,b]$, then $\int_a^b f \leq \int_a^b g$.

Proof. By the Theorems 1 and 2, the functions $-f$ and $g-f$ are Riemann integrable. Since $g(x) - f(x) \geq 0$ by hypothesis, we have by the above Theorems 1,2 and 3,

$$0 \leq \int_a^b (g-f) = \int_a^b [g+(-f)] = \int_a^b g + \int_a^b (-f) = \int_a^b g - \int_a^b f.$$

From this, we have $\int_a^b f \leq \int_a^b g$.

Corollary 2. If $f \in R[a,b]$, then $|f| \in R[a,b]$ and we have

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. Since $|f|$ is continuous at every point where $|f| \in R[a,b]$.

Since $f(x) \leq |f(x)| = |f|(x)$ for all $x \in [a,b]$, from Corollary 1 above, we get

$$\int_a^b f \leq \int_a^b |f|. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since $-f(x) < |f|(x)$ for all $x \in [a,b]$, we have again using Corollary1 above,

$$-\int_a^b f \leq \int_a^b |f|. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2), we get

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Note. In the following, we shall give the proof of the first part of the above Corollary 2 by using the definition of Riemann integration.

NOTES

Theorem 4. If $f \in R[a,b]$, then $|f| \in R[a,b]$.

Proof. Since f is bounded in $[a,b]$, $|f(x)| \leq k$ for every $x \in [a,b]$ so that $|f|$ is bounded. Let $\varepsilon > 0$ be given and let $P = a = x_0 < x_1 < \cdots x_{k-1} < x_k < \cdots x_n = b$ be a partition of $[a,b]$ and let $x, y \in I_k$.

Then we have the following

$$|[f(x)]| - |[f(y)]| \leq |f(x) - f(y)| \leq M[f; I_k] - m[f; I_k].$$

As x, y vary over I_k, we have from above,

$$M[|f|, I_k] - m[|f|, I_k] \leq M[f; I_k] - m[f; I_k]$$

This implies

$$U[|f|; I_k] - L[|f|; I_k] \leq U[f; I_k] - L[f; I_k] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since $f \in R[a,b]$, we get $U[f; I_k] - L[f; I_k] < \varepsilon$ for every $\varepsilon > 0$.

Using (1) in (2), we get $U[|f|; I_k] - L[|f|; I_k] < \varepsilon$.

Hence, $|f| \in R[a,b]$.

Note. The converse of the above theorem is not true and it is shown by the following example.

Let f be a real valued function defined on $[a,b]$ by

$$f(x) = \begin{cases} 1 \; when \; x \; is \; rational \\ -1 \; when \; x \; is \; irrational. \end{cases}$$

For any partition of $[a,b]$. We can check easily

$$\int_a^{\bar{b}} f = (b-a) \text{ and } \int_{\underline{a}}^{b} f = -(b-a).$$

This implies that f is not Riemann integrable in $[a,b]$. But $|f(x)| = 1$ for every $x \in [a,b]$. Hence, $|f|$ is Riemann integrable and its value equals to $(b-a)$.

Theorem 5. If $f \in R[a,b]$ and $a \leq c \leq b$, then

$$f \in R[a,c], f \in R[c,b] \text{ and } \int_a^b f = \int_a^c f + \int_c^b f.$$

NOTES

Proof. The set D of points $[a, b]$ at which f is not continuous is of measure zero. Let D_1 be the set of points of discontinuities of f in$[a, c]$. Then $D_1 \subset [a, c]$ is a subset of a set of measure zero and hence it is of measure zero. So $f \in R[a, c]$. Similarly $f \in R[c, b]$.

If P is any partition of $[a, c]$ and Q is any partition of $[c, b]$, then $P \cup Q$ is a partition of $[a, b]$ whose component intervals are those of P together with those of Q.

Hence, we have $L[f; P] + L[f; Q] = L[f; P \cup Q] \leq \int_{-a}^{b} f$

and so, we have $L[f; P] + L[f; Q] \leq \int_{a}^{b} f$.

By taking the least upper bound on the left over all P, keeping Q fixed, we obtain

$$\int_{a}^{c} f + L[f; Q] \leq \int_{a}^{b} f.$$

Now taking least upper bound over all Q, we get

$$\int_{a}^{c} f + \int_{c}^{b} f \leq \int_{a}^{b} f \qquad (1)$$

By using similar argument by considering the upper sums, we get the reverse inequality,

$$\int_{a}^{c} f + \int_{c}^{b} f \geq \int_{a}^{b} f \qquad (2)$$

From (1) and (2), we get $\int_{a}^{c} f + \int_{c}^{b} f = \int_{a}^{b} f$.

Theorem 6. If f is continuous on a closed and bounded interval $[a, b]$, if $f(x) \geq 0$ for $(a \leq x \leq b)$ and if $f(c) > 0$ for some $c \in [a, b]$, then $\int_{a}^{b} f(x) dx > 0$.

Proof. From the properties of continuous functions in Chapter 4, there exists a $\delta > 0$ such that $f(x) > \frac{1}{2} f(c)$ for some $x \in (c - \delta, c + \delta) \subset [a, b]$.

Now we have by Theorem 5,

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f = \int_{c+\delta}^b f.$$

The above formula can be suitably modified when we have $c - \delta < a$ or $c + \delta > b$. By Theorem 3 of the order preserving property of the integral, we get

$$\int_a^b f > 0 + \int_{c-\delta}^{c+\delta} \frac{1}{2} f(c)\ dx + 0 = 2\delta \frac{1}{2} f(c) = \delta f(c) > 0$$

This proves that $\int_a^b f > 0$.

Theorem 7. If f is continuous on $[a, b]$, $f(x) \geq 0$ for $a \leq x \leq b$, and if $\int_a^b f(x)dx = 0$, then f is identically zero on $[a, b]$.

Proof. By hypothesis $f(x) \geq 0$ in $[a, b]$. If f is not identically zero in $[a, b]$, there exists a point c in $[a, b]$ such that $f(c) > 0$.

Now f is a continuous function in the bounded closed interval $[a, b]$ and $f(x) \geq 0$. Since $f(c) > 0$ for $a, c \in [a, b]$, by the previous theorem, $\int f(x)dx > 0$ which contradicts the hypothesis. Hence f is identically zero on $[a, b]$.

Theorem 8. If f is continuous on $[a, b]$ and if $F(x) = \int_a^x f(t)dt$ for some $x \in (a, b)$, then F is continuous on $[a, b]$.

Proof. Let $x', x'' \in [a, b]$ with $x' > x''$.

Then $F(x') - F(x'') = \int_a^{x'} f(x)dx - \int_a^{x''} f(x)dx = \int_{x''}^{x'} f(x)dx$.

Now given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$ where M is the lub $f(x)$ in $[a, b]$.

Now $| F(x') - F(x'')| \leq M(x' - x'') < M\frac{\varepsilon}{M} = \varepsilon$.

Hence, $| F(x') - F(x'')| < \varepsilon$ whenever $|x' - x''| < \delta$.

This proves that F is continuous on $[a, b]$.

Theorem 9. If $f \in R[a, b]$, then the following statements are true.

i. $f \in R[c, d]$ for every subinterval $[c, d] \subset [a, b]$.
ii. $f^2 \in R[a, b]$.
iii. $f.g \in R[a, b]$, whenever $g \in R[a, b]$.

iv. If $f, g \in R[a,b]$, then $f/g \in R[a,b]$ where g is bounded away from zero.

v. If f and g are bounded functions having the same discontinuities on $[a,b]$, then $f \in R[a,b]$ if and only if $g \in R[a,b]$.

vi. Let $g \in R[a,b]$ and assume that $m \le g(x) \le M$ for all $x \in [a,b]$. If f is continuous on $[m,M]$, then the composite function defined by $h(x) = f[g(x)]$ is Riemann integrable on $[a,b]$.

Proof. (i). Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a,b]$ such that

$$U[f;P][a,b] - L[f;P][a,b] < \varepsilon.$$

Let $P^* = P \cup \{c,d\}$. The P^* is a refinement of $[a,b]$ and by the Theorem 2 of 8.1, we have

$$U[f;P^*][a,b] \le U[f;P][a,b] \text{ and } L[f;P^*][a,b] \ge L[f;P][a,b]$$

Now let $Q = P^* \cap [c,d]$. Then Q is obtained by restricting P^* to $[c,d]$. Hence we have the inequality,

$$U[f;Q][c,d] - L[f;Q][c,d] \le U[f;P^*][a,b] - L[f;P^*][a,b] \qquad (1)$$

because the left-hand side has fewer terms which are all non-negative than the right hand side. Since $f \in R[a,b]$, we get

$$U[f;P^*][a,b] - L[f;P^*][a,b] < \varepsilon. \qquad (2)$$

Using (2) in (1), we get $U[f;Q][c,d] - L[f;Q][c,d] < \varepsilon$.

Therefore, we get $f \in R[c,d]$.

(ii). Let $\varepsilon > 0$ be given and then there exists a partition P of $[a,b]$ such that

$$U[f;P] - L[f;P] < \varepsilon.$$

We know that $M[f^2; I_k] = M[\,|f|; I_k]^2$ and $m[f^2; I_k] = m[\,|f|; I_k]^2$

$$U[f^2;P] - L[f^2;P] = \sum_{k=1}^{n} [M[f^2;I_k] - m[f^2;I_k]]\ |I_k|$$

NOTES

$= \sum_{k=1}^{n}[M[\,|f|; I_k]^2 - m[\,|f|; I_k]^2]\,|I_k|$

$= \sum_{k=1}^{n}\{M[\,|f|; I_k] + m[\,|f|; I_k]\}\{M[\,|f|; I_k] - m[\,|f|; I_k]\}|I_k|$

$\leq 2\,\lambda\,\sum_{k=1}^{n}\{M[\,|f|; I_k] - m[\,|f|; I_k]\}|I_k|$

where λ is an upper bound of f in $[a, b]$. Therefore, we have

$$U\left[f^2; P\right] - L\left[f^2; P\right] < 2\,\lambda\,[U[|f|; P] - L[|f|; P]]$$

Hence, by using theorem 4, we get

$$U[|f|; P] - L[|f|; P] < \frac{\varepsilon}{2\lambda}$$

Hence, $U\left[f^2; P\right] - L\left[f^2; P\right] < \varepsilon$ and therefore, $f^2 \in R[a, b]$.

(iii) This follows from the following identity and Theorem 2 (ii) proved above.

$$2\,f(x)g(x) = [f(x) + g(x)]^2 - [f(x)]^2 - [g(x)]^2$$

(v) Since $g(x) \neq 0$ for any $x \in [a, b]$, applying (iii), $f.\frac{1}{g} \in R[a, b]$,

provided $\frac{1}{g} \in R[a, b]$ whenever $g \in R[a, b]$ under the given condition.

Hence we shall prove that $\frac{1}{g} \in R[a, b]$, whenever $g \in R[a, b]$ and g is bounded away from zero.

By hypothesis, g is bounded away from zero and so we have

$$|g(x)| > k \text{ for every } x \in [a, b].$$

Let $P: a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$ and let $\alpha, \beta \in I_k$.

$$\left|\frac{1}{g(\alpha)} - \frac{1}{g(\beta)}\right| = \left|\frac{g(\beta)-g(\alpha)}{g(\alpha)g(\beta)}\right| < \frac{1}{k^2}|g(\beta) - g(\alpha)|.$$

From this, we get

$$M\left[\frac{1}{g};\ I_k\right] - m\left[\frac{1}{g};\ I_k\right] < \frac{1}{k^2}[M[g; I_k] - m[g; I_k]].$$

This implies $U\left[\frac{1}{g};\ P\right]-L\left[\frac{1}{g};\ P\right]<\frac{1}{k^2}[U[g;P]-L[g;P]]$ (3)

Since $g\in R[a,b]$, given $\varepsilon>0$, there exists a partition P such that

$U[g;P]-L[g;P]<k^2\varepsilon.$ (4)

Using (4) in (3), we get $U\left[\frac{1}{g};\ P\right]-L\left[\frac{1}{g};\ P\right]<\varepsilon.$

(v) This follows by Theorem 2.

(vi) Since h is uniformly continuous on $[m,M]$, given $\varepsilon>0$, there exists a $\delta>0$ such that $\delta<\varepsilon$ and

$|f(s)-f(t)|<\varepsilon$, if $|s-t|<\delta$ and $s,t\ \in[m,M]$.

Since $g\in R[a,b]$, there is a partition, $P\colon a=x_0,x_1,x_2,\dots,x_n=b$ of $[a,b]$ such that

$U[g;P]-L[g;P]<\delta^2.$

Let $M^*[h;I_k]$ and $m^*[h;I_k]$ for h corresponding to $M[g;I_k]$ and $m[g;I_k]$ on $[a,b]$. Divide the numbers $1,2,3,\dots,n$ into two groups such that $k\in A$ if $M[g;I_k]-m[g;I_k]<\delta$ and $k\in B$ if $M[g;I_k]-m[g;I_k]\geq\delta$.

Hence, if $k\in A$, our choice of δ shows that

$$M^*[h;I_k]-m^*[h;I_k]<\varepsilon$$

Let $\lambda=lub\ |h(t)|$ in $m<t<M$. Then if $k\in B$, we get

$$M^*[h;I_k]-m^*[h;I_k]<2\lambda.$$

Now, $\delta\sum_{k\in B}|I_k|<\sum_{k\in B}\big[M[g;I_k]-m[g;I_k]\big]|I_k|<\delta^2.$

From this, it follows that $\sum_{k\in B}|I_k|<\delta,$

$$U[h;P]-L[h;P]=\sum_{k=1}^{n}M^*[h;I_k]-m^*[h;I_k]\ |I_k|$$

$\leq\ \sum_{k\in A}M^*[h;I_k]-m^*[h;I_k]\ |I_k|+\sum_{k\in B}M^*[h;I_k]-m^*[h;I_k]\ |I_k|$

$$<\ \varepsilon[b-a]+2\lambda\delta$$

NOTES

$$< \varepsilon[b-a] + 2\lambda\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $h \in R[a,b]$.

10.5 THE FUNDAMENTAL THEOREM OF CALCULUS

In the previous chapter, we have established that there are real valued functions which are not the derivatives of any function on $[-1,1]$. In the following theorem, we shall establish that if f is continuous on $[a,b]$, there exists a function F on $[a,b]$ such that $F'(x) = f(x)$, thus establishing the link between the concepts of derivative and integral.

Theorem 1. (First Fundamental Theorem of Calculus). If f is continuous on the closed bounded interval $[a,b]$ and if

$F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for $a \le x \le b$.

Proof. For any fixed $x \in [a,b]$, choose $h \neq 0$ and $x+h \in [a,b]$. Then we have the following:

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt.$$

We can rewrite the above step as,

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt. \qquad (1)$$

Since f is continuous on the closed and bounded interval $[x, x+h]$, we know that f attains a maximum value M and a minimum value m at points of $[x, x+h]$ by Intermediate Value Theorem of continuous functions. Hence, there exist points $t_1, t_2 \in [x, x+h]$ such that $f(t_1) = m, f(t_2) = M$ and $m \le f(t) \le M$. So we get

$$\int_x^{x+h} m\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} M\,dt \qquad (2)$$

But $\int_x^{x+h} m\,dt = hm$ and $\int_x^{x+h} M\,dt = M\,h$ (3)

Therefore, using (3) in (2) we get

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh.$$

So, we can find a θ such that $m < \theta < M$ and

$$\theta = \frac{1}{h}\int_{x}^{x+h} f(t)\,dt$$

Since f takes every value between m and M by Intermediate Value Theorem of continuous functions, there must exists a point $c(h)$ in $[x, x+h)$ such that $f[c(h)] = \theta$. Thus we have proved that if $h > 0$, there exists $c(h)$ in $[x, x+h]$ such that

$F[c(h)] = \frac{1}{h}\int_{x}^{x+h} f(t)\,dt$

From (1), we get $\frac{F(x+h)-F(x)}{h} = f[c(h)]$

Since $x \le c(h) \le x+h$, we have $\lim_{h\to 0} c(h) = x$.

Since f is continuous at x, the right side of (4) has the limit $f(x)$. Hence, the left side of (4) approaches $F'(x)$ as $h \to 0$. So we get

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x).$$

In the above proof, we have assumed that h is positive. If h is negative, we take $[x+h, x]$ instead of $[x, x+h]$ and make suitable modification in the proof.

Note. The continuity of f is only a sufficient condition for a function to be a derivative of a function on $[a, b]$. The continuity is not a necessary condition as shown in the Example 7 of 7.1.

Instead of assuming continuity throughout $[a, b]$, we assume f to be continuous at any point x of $[a, b]$ and $f \in R[a, b]$. Under this hypothesis, we have the following theorem.

Theorem 2. If $f \in R[a, b]$, if $F(x) = \int_a^x f(t)\,dt$ where $a \le x \le b$ and if f is continuous at $x_0 \in [a, b]$, then $F'(x_0) = f(x_0)$.

Proof. For $h > 0$, let $I(h)$ denote the interval $[x_0, x_0 + h]$. If $\omega[f; I(h)]$ is the oscillation of f in $I(h)$, we have for $t \in I(h)$, $|f(t) - f(x_0)| \le \omega[f; I(h)]$. So

$$f(x_0) - \omega[f; I(h)] \le f(t) \le f(x_0) + \omega[f; I(h)]$$

where $t \in I(h)$.

Hence, we have

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$$h[f(x_0) - \omega[f; I(h)]] \leq \int_{x_0}^{x_0+h} f(t)dt \leq h[f(x_0) + \omega[f; I(h)]].$$

Since $h > 0$, we have after dividing by h,

$$f(x_0) - \omega[f; I(h)] \leq \frac{F(x_0+h)-F(x_0)}{h} \leq f(x_0) + \omega[f; I(h)] \qquad (1)$$

Since f is continuous at x_0 by hypothesis, we have

$$\lim_{h\to 0} \omega[f; I(h)] = 0 \qquad (2)$$

Taking the limit as $h \to 0$ in (1) and using (2), we get

$$F'(x_0) = f(x_0).$$

In the above, (1) is established for $h > 0$. We can establish (1) if $h < 0$ in a similar manner.

DEFINITION 1. A function F is called a primitive or an antiderivative of a function f on a bounded closed interval $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$.

The First Fundamental Theorem of calculus states that we can always construct a primitive of a continuous function by integration.

Now we shall prove the second fundamental theorem of calculus establishing integration as the anti-differentiation or reverse process of differentiation.

THEOREM 3. (Second Fundamental Theorem of Calculus). If f is a continuous function on the closed bounded interval $[a, b]$ and if $\phi'(x) = f(x)$ for $x \in [a, b]$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

Proof. Let $F(x) = \int_a^x f(t)\, dt$. Since f is continuous, by the First Fundamental Theorem, we have

$$F'(x) = f(x) \text{ for } a \leq x \leq b. \qquad (1)$$

By hypothesis, $\phi'(x) = f(x)$. Hence, we have $F'(x) = \phi'(x)$ for all $x \in [a, b]$. Hence, by the Theorem 4 of 7.3, $F(x) = \phi(x) + c$ for $a \leq x \leq b$ and for some constant c in R.

Hence, $\quad F(b) - F(a) = [\phi(b) + c] - [\phi(a) + c] = \phi(b) - \phi(a).$

But $\quad F(a) = \int_a^a f(t)\, dt = 0$ from the definition.

Thus, $F(b) = \phi(b) - \phi(a)$. Since $F(b) = \int_a^b f(t)\,dt$, we have

$\int_a^b f(t)\,dt = \phi(b) - \phi(a)$.

Example 1. If $(x) = 3\sin x + 2e^x$, find the primitive F of f and use the Second Fundamental Theorem to evaluate $\int_a^b f(x)\,dx$.

Now let $F(x) = 2e^x - 3\cos x$ whose derivative is the given function f in any bounded closed interval $[a, b]$. Since $\sin x$ and e^x are continuous in $[a, b]$, $3\sin x + 2e^x$ is continuous in $[a, b]$. Hence, by the Second Fundamental Theorem of Calculus, we get

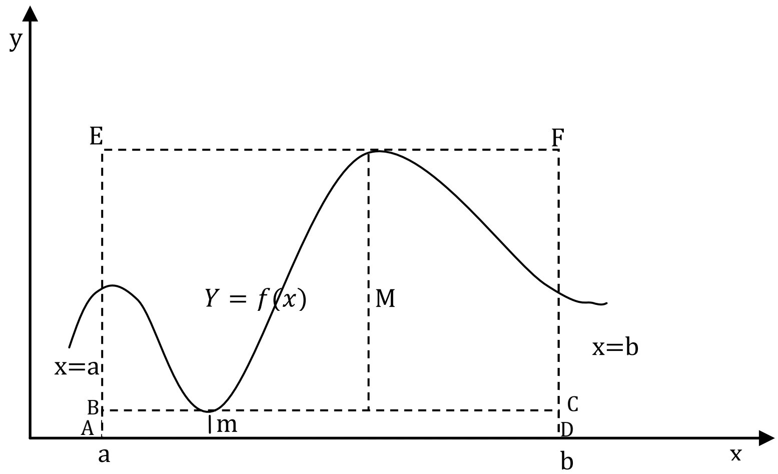
$\int_a^b (3\sin x + 2e^x)\,dx = F(b) - F(a) = 2(e^b - e^a) - 3(\cos b - \cos a)$.

Theorem 4. If $f(x)$ is continuous in the bounded closed interval $[a, b]$ then there exists a number c lying between a and b such that $\int_a^b f(x)\,dx = (b-a)f(c)$.

Note: Recharge $ABCD \leq \int_a^b f(x)\,dx \leq$ Recharge $AEFD$

Proof. let $b > a$. We can rewrite the above inequality as

$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$.



So $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a value between m and M of a continuous function on $[a, b]$. Therefore, by Intermediate Value Theorem for continuous function in $[a, b]$, f takes this value at some point c of $[a, b]$. So we get

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$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ for some c in $[a,b]$.

This proves that $\int_a^b f(x)\,dx = (b-a)f(c)$.

Check your progress

1. Define the measure zero set
2. State First Fundamental Theorem of Calculus.
3. Define primitive
4. State Second Fundamental Theorem of Calculus.

10.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1) A subset E of R is said to be of measure zero if for each $\varepsilon > 0$, there exists a finite or countable number of open intervals (I_n) such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_n |I_n| < \varepsilon$.

2) If f is continuous on the closed bounded interval $[a,b]$ and if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for $a \le x \le b$.

3) A function F is called a primitive or an antiderivative of a function f on a bounded closed interval $[a,b]$ if $F'(x) = f(x)$ for all x in $[a,b]$.

4) If f is a continuous function on the closed bounded interval $[a,b]$ and if $\phi'(x) = f(x)$ for $x \in [a,b]$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

10.7 SUMMARY

- A subset E of R is said to be of **measure zero** if for each $\varepsilon > 0$, there exists a finite or countable number of open intervals (I_n) such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_n |I_n| < \varepsilon$.
- If each of $E_1, E_2, \ldots,$ of R is of measure zero, then $\bigcup_{n=1}^{\infty} E_n$ is also of measure zero.
- Every countable set of R is a set of measure zero.
- If a property is true on $[a,b]$ expect on a set of measure zero, then the property is said to be **true almost everywhere** on $[a,b]$ or for almost all points of $[a,b]$. That is, the set of points of $[a,b]$ at which the property is not true is a set of measure zero.
- Let f be a bounded function on the closed and bounded interval $[a,b]$. Then f is Riemann integrable if and only if f is continuous at almost every point in $[a,b]$.

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- If f is Riemann integrable on $[a, b]$ and c is any real number, then cf is Riemann integrable and
$$\int_a^b cf = c\int_a^b f.$$
- If $f \in R[a, b]$ and $g \in R[a, b]$, then $f + g \in R[a, b]$ and
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$
- If $f \in R[a, b]$ and if $f(x) \geq 0$ almost everywhere on $[a, b]$, then
$$\int_a^b f \geq 0.$$
- $f \in R[a, b]$ and $g \in R[a, b]$ and if $f(x) \leq g(x)$ almost everywhere on $[a, b]$, then $\int_a^b f \leq \int_a^b g$.
- If $f \in R[a, b]$, then $|f| \in R[a, b]$ and we have
$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$
- If $f \in R[a, b]$, then $|f| \in R[a, b]$.
- If $f \in R[a, b]$ and $a \leq c \leq b$, then
  $f \in R[a, c], f \in R[c, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.
  - If f is continuous on a closed and bounded interval $[a, b]$, if $f(x) \geq 0$ for $(a \leq x \leq b)$ and if $f(c) > 0$ for some $c \in [a, b]$, then $\int_a^b f(x)dx > 0$.
- If f is continuous on $[a, b]$, $f(x) \geq 0$ for $a \leq x \leq b$, and if $\int_a^b f(x)dx = 0$, then f is identically zero on $[a, b]$.
- If f is continuous on $[a, b]$ and if $F(x) = \int_a^x f(t)dt$ for some $x \in (a, b)$, then F is continuous on $[a, b]$.
- **First Fundamental Theorem of Calculus:** If f is continuous on the closed bounded interval $[a, b]$ and if
$F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for $a \leq x \leq b$.
- If $f \in R[a, b]$, if $F(x) = \int_a^x f(t)\,dt$ where $a \leq x \leq b$ and if f is continuous at $x_0 \in [a, b]$, then $F'(x_0) = f(x_0)$.
- A function F is called a **primitive** or an **antiderivative** of a function f on a bounded closed interval $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$.
- **Second Fundamental Theorem of Calculus**: If f is a continuous function on the closed bounded interval $[a, b]$ and if $\phi'(x) = f(x)$ for $x \in [a, b]$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

- If $f(x)$ is continuous in the bounded closed interval $[a, b]$ then there exists a number c lying between a and b such that $\int_a^b f(x)\,dx = (b-a)f(c)$.

10.8 KEYWORDS

- A subset E of R is said to be of **measure zero** if for each $\varepsilon > 0$, there exists a finite or countable number of open intervals (I_n) such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_n |I_n| < \varepsilon$.
- If a property is true on $[a, b]$ expect on a set of measure zero, then the property is said to be **true almost everywhere** on $[a, b]$ or for almost all points of $[a, b]$. That is, the set of points of $[a, b]$ at which the property is not true is a set of measure zero.
- A function F is called a **primitive** or an **antiderivative** of a function f on a bounded closed interval $[a, b]$ if $F'(x) = f(x)$ for all x in $[a, b]$.

10.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

1) Prove that if f is continuous on $[0,1]$ and if $g(x) = f(x)$ almost everywhere $x \in [0,1]$, then g is continuous almost everywhere in $[0,1]$.

10.10 FURTHER READINGS

1) Arumugam & Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2010.
2) Richard R. Goldbrg, Methods of Real Analysis, Oxford & IBH Publishing Company, New Delhi.
3) D. Somasundaram & B. Choudhary, A first course in Mathematical Analysis, Narosa Publishing House, Chennai.
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UNIT 11 CONTRACTION MAPPING AND ITS APPLICATIONS

Structure

11.0 INTRODUCTION

In this chapter we introduce a class of functions called contraction mappings and we prove a simple result regarding contraction mappings on a complete metric space. We illustrate the use of this theorem in classical analysis by proving the existence and uniqueness of solution of a differential equation of first order.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by Contraction Mapping
- Discuss the applications of contraction mapping

11.2 CONTRACTION MAPPING

11.2.1 Definition and Examples

Definition: Let (M, d) be a metric space. A mapping $T: M \to M$ is called a *contraction mapping* if there exists a positive real number $\alpha < 1$ such that $d\big(T(x), T(y)\big) \leq \alpha\, d(x, y)$ for all $x, y \in M$.

Note: If T is a contraction mapping then the distance $d\big(T(x),T(y)\big)$ is less than the distance $d(x,y)$. Thus applying T to any two points x,y contracts the distance between the two points.

Example 1. $T:\left[0,\frac{1}{3}\right] \to \left[0,\frac{1}{3}\right]$ defined by $T(x) = x^2$ is a contraction mapping.

Proof. Let $x,y \in \left[0,\frac{1}{3}\right]$.

$$\text{Then } d\big(T(x),T(y)\big) = |x^2 - y^2|$$

$$= |x+y||x-y$$

$$\leq \frac{2}{3}|x-y| \ \ (\text{since } x,y \leq \frac{1}{3})$$

$$= \frac{2}{3}d(x,y)$$

$$\therefore d(T(x),T(y)) \leq \frac{2}{3}d(x,y)$$

Hence T is a contraction mapping.

Example 2. $T: R \to R$ defined by $T(x) = \frac{1}{2}x$ is a contraction mapping since $d\big(T(x),T(y)\big) = \frac{1}{2}d(x,y)$.

Example 3. $T: l_2 \to l_2$ defined by $T(x) = \left(\frac{1}{2}x_n\right)$ is a contraction mapping where $x = (x_n)$.

Proof. Let $x,y \in l_2$. Let $x = (x_n), y = (y_n)$.

$$\text{Now, } d\big(T(x),T(y)\big) = \left[\sum_{n=1}^{\infty}\left(\frac{x_n}{2} - \frac{y_n}{2}\right)^2\right]^{1/2} = \frac{1}{2}\left[\sum_{n=1}^{\infty}(x_n - y_n)^2\right]^{1/2}$$

$$= \frac{1}{2}d(x,y).$$

$\therefore$ T is a contraction mapping.

Example 4. Let $T: [0,1] \to [0,1]$ be a differentiable function. If there is a real number α with $0 < \alpha < 1$ such that $|T'(x)| \leq \alpha$ for all $x \in [0,1]$ where T' is the derivative of T then T is a contraction mapping.

Proof. Let $x,y \in [0,1]$ and $x < y$.

By mean value theorem $T(y) - T(x) = (y,x)T'(z)$ where $x < z < y$.

$\therefore |T(y) - T(x)| = |y - x|\,|T'(z)| \le \alpha |y - x|.$

$$\therefore d\big(T(y), T(x)\big) \le \alpha\, d(y, x) \text{ and } 0 < \alpha < 1.$$

$\therefore T$ is a contraction mapping.

11.3 CONTRACTION MAPPING THEOREM AND ITS APPLIVATIONS

Theorem 11.1 Let $T: M \to M$ be a contraction mapping. Then T is continuous on M.

Proof. Since T is contraction mapping

$$d\big(T(x), T(y)\big) < d(x, y) \text{ for all } x, y \in M \text{.........(1)}$$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$.

$$\text{Then } d(x, y) < \delta \Longrightarrow d\big(T(x), T(y)\big) < \varepsilon \quad (\text{ by (1)})$$

$$\therefore T \text{ is continuous.}$$

Theorem 11.2 (Contraction mapping theorem)

Let (M, d) be a complete metric space. Let $T: M \to M$ be a contraction mapping. Then there exists a unique point $x \in M$ such that $T(x) = x$.

(i.e.) T has exactly one fixed point.

Proof. Let $x_0$be an arbitrary point in M.

Let
$$x_1 = T(x_0)$$
$$x_2 = T(x_1)$$
$$x_3 = T(x_2)$$
$$\dots \dots \dots \dots$$
$$\dots \dots \dots \dots$$
$$x_n = T(x_{n-1})$$
$$\dots \dots \dots \dots$$

We claim that (x_n) is a Cauchy sequence in M. Since T is a contraction mapping, there exists a real number α such that $0 < \alpha < 1$ and $d\big(T(x), T(y)\big) \le \alpha\, d(x, y)$.

$$\therefore d(x_n, x_{n+1}) = d\big(T(x_{n-1}), T(x_n)\big)$$

$$\leq \alpha\, d(x_{n-1}, x_n)$$

$$\leq \alpha^2 d(x_{n-2}, x_{n-1})$$

$$\leq \alpha^3 d(x_{n-3}, x_{n-2})$$

… … … … … … … … …

… … … … … … … … …

$$\leq \alpha^n d(x_0, x_1)$$

$$\therefore d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1) \text{ .... } (1)$$

Now, let $m, n \in N$ and $m > n$.

Then $\quad d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$

$$\leq \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \cdots$$

$$+\alpha^{m-1} d(x_0, x_1) \text{ using } (1)$$

$$= \alpha^n d(x_0, x_1)[1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}]$$

$$< \alpha^n d(x_0, x_1)\left[\frac{1}{1-\alpha}\right].$$

Thus $d(x_n, x_m) < \frac{\alpha^n d(x_0,x_1)}{1-\alpha}$ for all m, n such that $m > n$.

Now, since $0 < \alpha < 1$, the sequence $(\alpha^n) \to 0$.

$\therefore$ Given $\varepsilon > 0$ there exists a positive integer n_1 such that $\frac{\alpha^n d(x_0,x_1)}{1-\alpha} < \varepsilon$ for all $n \geq n_1$.

Then $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_1$.

Hence (x_n) is a Cauchy sequence in M.

Since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Also by Theorem, T is continuous and hence $(T(x)) \to T(x)$.

$$\therefore T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Thus $T(x) = x$.

Hence x is a fixed point of T.

Now, Suppose there exists $y \in M$ such that $y \neq x$ and $T(y) = y$.

Then $d(x,y) = d(T(x), T(y)) \leq \alpha d(x,y)$.

$\therefore d(x,y)(1-\alpha) \leq 0$.

But $d(x,y) > 0$ and $1-\alpha > 0$ which is a contraction.

$\therefore x$ is the unique fixed point of T.

Theorem 11.3 (Picard's Theorem)

Let $\frac{dy}{dx} = f(x,y)$ be a given differential equation where $f(x,y)$ is continuous in a closed rectangle

$$F = \{(x,y)/a_1 \leq x \leq a_2 \ \ and \ b_1 \leq x \leq b_2 \}$$

and satisfy the Lipchitz condition given by $|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|$ for all (x,y_1) and $(x,y_2) \in F$. Let (x_0,y_0) be an interior point of F. Then there exists a unique solution $y = \varphi(x)$ of the differential equation such that $\varphi(x_0) = y_0$.

Proof. We first replace our problem by an equivalent problem relating to an integral equation.

Let $y = \varphi(x)$ be a solution to the given differential equation such that $\varphi(x_0) = y_0$.

Then $\frac{d}{dx}\big(\varphi(x)\big) = f(x, \varphi(x))$.

Integrating from x_0 to x we get $\varphi(x) - \varphi(x_0) = \int_{x_0}^{x} f\big(t, \varphi(t)\big)dt$.

(i.e.) $\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt$..............................(1)

Now, if $y = \varphi(x)$ satisfies the integral equation (1), then it satisfies the given differential equation and $\varphi(x_0) = y_0$.

$\therefore$ It is enough to prove that the integral equation (1) has a unique solution.

Now, since f is continuous on the compact set F, it is bounded.

$\therefore$ There exist a real number $k > 0$ such that
$|f(x,y)| \leq k$ for all $(x,y) \in F$................................(2)

Now, choose a real number $\delta > 0$ such that $M\delta < 1$ and a rectangle
$F_1 = \{(x,y)/|x - x_0| \leq \delta \ \ and \ |y - y_0| \leq k\delta\}$ contained in F.

Let C^* be the set of all continuous functions
$\varphi \in C[x_0 - \delta, x_0 + \delta]$ such that $|\varphi(x) - y_0| \leq k\delta$.

By solved problem, C^* is a complete metric space.

Let $\varphi \in C^*$.

Define $T(\varphi) = \psi$ where $\psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt$.

Clearly ψ is continuous.

Also, $|\psi(x) - y_0| = \left|\int_{x_0}^{x} f\big(t, \varphi(t)\big)dt\right|$. (from 1)

$$\leq k(x - x_0)$$

$$\leq k\delta$$

$\therefore |\psi(x) - y_0| \leq k\delta.$

$$\therefore \psi \in C^*$$

$\therefore T$ is a mapping from $C^* \to C^*$.

Now we claim that T is a contraction mapping.

Let $\varphi_1, \varphi_2 \in C^*, T(\varphi_1) = \psi_1$ and $T(\varphi_2) = \psi_2$. Then

$$|\psi_1(x) - \psi_2(x)| = \left|\int_{x_0}^{x} |f\big(t, \varphi_1(t)\big) - f\big(t, \varphi_2(t)\big)|dt\right|$$

$$\leq \int_{x_0}^{x} |f\big(t, \varphi_1(t)\big) - g\big(t, \varphi_2(t)\big)|dt$$

$$\leq M \int_{x_0}^{x} |\varphi_1(t) - \varphi_2(t)|dt$$

(using Lipchitz's condition)

$$\leq M(x, x_0) \sup\{|\varphi_1(t) - \varphi_2(t)|\}$$

$$\leq M\delta d(\varphi_1, \varphi_2).$$

Thus $|\psi_1(x) - \psi_2(x)| \leq M\delta d(\varphi_1, \varphi_2)$.

$$\therefore d(T(\varphi_1), T(\varphi_2))| \leq M\delta d(\varphi_1, \varphi_2).$$

Since $M\delta < 1$ T is a contraction mapping.

Hence there exists a unique function $\varphi \in C^*$ such that $T(\varphi) = \varphi$.

$$\therefore \varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt.$$

$\therefore$ φ is the unique solution of the integral equation (1). Hence the theorem.

Check your progress
1) Define contraction mapping
2) If $T: M \to M$ is a contraction mapping. Then T?
3) State contraction mapping theorem.

11.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1) Let (M, d) be a metric space. A mapping $T: M \to M$ is called a *contraction mapping* if there exists a positive real number $\alpha < 1$ such that $d\big(T(x), T(y)\big) \leq \alpha\, d(x, y)$ for all $x, y \in M$.
2) Let $T: M \to M$ be a contraction mapping. Then T is continuous on M.
3) Let (M, d) be a complete metric space. Let $T: M \to M$ be a contraction mapping. Then there exists a unique point $x \in M$ such that $T(x) = x$.

(i.e.) T has exactly one fixed point.

11.5 SUMMARY

- Let (M, d) be a metric space. A mapping $T: M \to M$ is called a ***contraction mapping*** if there exists a positive real number $\alpha < 1$ such that $d\big(T(x), T(y)\big) \leq \alpha\, d(x, y)$ for all $x, y \in M$.
- Let $T: M \to M$ be a contraction mapping. Then T is continuous on M.
- **Contraction mapping theorem:** Let (M, d) be a complete metric space. Let $T: M \to M$ be a contraction mapping. Then there exists a unique point $x \in M$ such that $T(x) = x$.

  (i.e.) T has exactly one fixed point.
- **Picard's Theorem:** Let $\frac{dy}{dx} = f(x, y)$ be a given differential equation where $f(x, y)$ is continuous in a closed rectangle

$$F = \{(x, y)/a_1 \leq x \leq a_2 \ \ and \ b_1 \leq x \leq b_2 \}$$

and satisfy the Lipchitz condition given by $|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$ for all (x, y_1) and $(x, y_2) \in F$. Let (x_0, y_0) be an interior point of F. Then there exists a unique solution $y = \varphi(x)$ of the differential equation such that $\varphi(x_0) = y_0$.

NOTES

11.6 KEYWORDS

Contraction mapping : Let (M, d) be a metric space. A mapping $T: M \to M$ is called a *contraction mapping* if there exists a positive real number $\alpha < 1$ such that $d\big(T(x), T(y)\big) \le \alpha\, d(x, y)$ for all $x, y \in M$.

Contraction mapping theorem: Let (M, d) be a complete metric space. Let $T: M \to M$ be a contraction mapping. Then there exists a unique point $x \in M$ such that $T(x) = x$.

(i.e.) T has exactly one fixed point.

11.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Prove that any contraction mapping T defined on a metric space is continuous.
2. Prove that any contraction mapping T defined on a metric space is uniformly continuous.
3. Prove that any contraction mapping T defined on a complete metric space has a unique fixed point.

11.8 FURTHER READINGS

1) Arumugam & Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2010.
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BLOCK IV
CONNECTED AND COMPACT METRIC SPACES

UNIT-12 CONNECTEDNESS

Structure

12.0 INTRODUCTION

In R consider the subsets $A = [1,2]$ and $B = [1,2] \cup [3,4]$. The set A consists of a single 'piece' whereas B consists of 'two pieces'. We say that A is a connected set and B is not a connected set. This intuitive idea is made precise in the following definition.

12.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is Connected sets
- Discuss Connectedness and Continuity

12.2 DEFINITION AND EXAMPLES

Definition. Let (M, d) be a metric space. M is said to be *connected* if M cannot be represented as the union of two disjoint non-empty open sets.

NOTES

If M is not connected it is said to be *disconnected*.

Example 1. Let $M = [1,2] \cup [3,4]$ with usual metric. Then M is disconnected.

Proof. $[1,2]$ and $[3,4]$ are open in M.

Thus M is the union of two disjoint non-empty open sets namely $[1,2]$ and $[3,4]$.

Hence M is disconnected.

Example 2. Any discrete metric space M with more than one point is disconnected.

Proof. Let A be a proper non-empty subset of M. Since M has more than one point such a set exists.

Then A^c is also non-empty.

Since M is discrete every subset of M is open.

$\therefore A$ and A^c are open.

Thus $M = A \cup A^c$ where A and A^c are two disjoint non-empty open sets.

$\therefore M$ is not connected.

Theorem 12.1 Let (M, d) be a metric space. Then the following are equivalent.

i. M is connected.

ii. M cannot be written as the union of two disjoint non-empty closed sets.

iii. M cannot be written as the union of two non-empty sets A and B such that $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

iv. M and $\emptyset$ are the only sets which are both open and closed in M.

Proof. (i) → (ii)

Suppose (ii) is not true.

$\therefore M = A \cup B$ where A and B are closed $A \neq \emptyset, B \neq \emptyset$ and $A \cap B = \emptyset$.

$$\therefore A^c = B \text{ and } B^c = A$$

NOTES

Since A and B are closed, A^c and B^c are open.

$\therefore$ B and A are open.

Thus M is the union of two disjoint non-empty open sets.

$\therefore$ M is not connected which is a contradiction.

$\therefore$ **(i)→(ii)**

(ii)→(iii)

Suppose (iii) is not true.

Then $M = A \cup B$ where $A \neq \emptyset, B \neq \emptyset$ and $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

We claim that A and B are closed.

Let $x \in \bar{A}$.

$\therefore x \notin B$ (Since $\bar{A} \cap B = \emptyset$)

$\therefore x \in A$ (Since $A \cap B = M$).

$\therefore A = \bar{A}$ and hence A is closed.

Similarly B is closed.

Now $A \cap B = \bar{A} \cap B$ (Since $A = \bar{A}$).

$= \emptyset$.

Thus $M = A \cup B$ where $A \neq \emptyset, B \neq \emptyset$, A and B are closed and $A \cap B = \emptyset$ which is a contradiction to (ii).

$\therefore$ **(ii)** $\rightarrow$ **(iii)**

(iii) $\rightarrow$ **(iv)**

Suppose (iv) is not true.

Then there exists $A \subseteq M$ such that $A \neq M$ and $A \neq \emptyset$ and A is both open and closed.

Let $B = A^c$.

Then B is also both open and closed and $B \neq \emptyset$.

Also $M = A \cup B$.

Further $\bar{A} \cap B = A \cap A^c$ (Since $\bar{A} = A$ and $B = A^c$).

NOTES

$$= \emptyset.$$

Similarly $A \cap \bar{B} = \emptyset$.

$\therefore M = A \cup B$ where $A \cap \bar{B} = \emptyset = \bar{A} \cap B$ which is a contradiction to (iii).

$$\therefore \textbf{(iii)} \rightarrow \textbf{(iv)}.$$

(iv)→(i)

Suppose M is not connected.

$\therefore\ M = A \cup B$ where $A \neq \emptyset, B \neq \emptyset$, A and B are open and $A \cap B = \emptyset$.

Then $B^c = A$. Now, since B is open A is closed.

Also $A \neq \emptyset$ and $A \neq M$ (Since $B \neq \emptyset$).

$\therefore A$ is a proper non-empty subset of M which is both open and closed which is a contraction to (iv).

$$\therefore \textbf{(iv)} \rightarrow \textbf{(i)}$$

The following theorem gives another equivalent characterization for connectedness.

Theorem 12.2 A metric space M is connected iff there does not exist a continuous function f from M onto the discrete metric space $\{0,1\}$.

Proof. Suppose there exists a continuous function f from M onto the discrete metric space $\{0,1\}$.

Since $\{0,1\}$ is discrete, $\{0\}$ and $\{1\}$ are open.

$\therefore A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are open in M.

Since f is onto, A and B are non-empty.

Clearly $A \cap B = \emptyset$ and $A \cup B = M$.

Thus $A \cup B = M$ where A and B are disjoint non-empty open sets.

$\therefore M$ is not connected which is a contradiction.

Hence there does not exist a continuous function from onto the discrete metric space $\{0,1\}$.

Conversely, suppose M is not connected.

NOTES

Then there exist disjoint non-empty open sets A and B in M such that $M = A \cup B$.

Now, define $f: M \to \{0,1\}$ by $f(x) = \begin{cases} 0 \text{ } if \text{ } x \in A \\ 1 \text{ } if \text{ } x \in B \end{cases}$

Clearly, f is onto.

Also $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$ and $f^{-1}(\{0,1\}) = M$.

Thus the inverse image of every open set in $\{0,1\}$ is open in M.

Hence f is continuous.

Thus there exists a continuous function f from M onto $\{0,1\}$ which is a contradiction. Hence M is connected.

Note. The above theorem can be restated as follows.

M is connected iff every continuous function $f: M \to \{0,1\}$ is not onto.

Solved Problems

Problem 1. Let M be a metric space. Let A be a connected subset of M. If B is a subset of M such that $A \subseteq B \subseteq \bar{A}$ then B is connected. In particular $\bar{A}$ is connected.

Solution. Suppose B is not connected.

Then $B = B_1 \cup B_2$ where $B_1 \neq \emptyset, B_2 \neq \emptyset, B_1 \cap B_2 = \emptyset$ and B_1 and B_2 are open in B.

Now, since B_1 and B_2 are open sets in B there exist open sets G_1 and G_2 in M such that $B_1 = G_1 \cap B$ and $B_2 = G_2 \cap B$.

$$\therefore B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B.$$

$$\therefore B \subseteq G_1 \cup G_2.$$

$\therefore A \subseteq G_1 \cup G_2$ (since $A \subseteq B$).

$$\therefore A \subseteq (G_1 \cup G_2) \cap A.$$

$$= (G_1 \cap A) \cup (G_2 \cap A)$$

Now, $(G_1 \cap A) and (G_2 \cap A)$ are open in A.

Further, $(G_1 \cap A) \cup (G_2 \cap A) = (G_1 \cup G_2) \cap A$.

NOTES

$$= (G_1 \cup G_2) \cap B \text{ (since } A \subseteq B)$$

$$= (G_1 \cap B) \cup (G_2 \cap B)$$

$$= B_1 \cap B_2$$

$$= \emptyset.$$

$\therefore (G_1 \cap A) \cup (G_2 \cap A) = \emptyset.$

Now, since A is connected, either $G_1 \cap A = \emptyset$ or $G_2 \cap A = \emptyset$.

Without lose of generality let us assume that $G_1 \cap \bar{A} = \emptyset$.

Since G_1 is open in M, we have $G_1 \cap \bar{A} = \emptyset$.

$$\therefore G_1 \cap B = \emptyset \qquad \text{(since } B \subseteq \bar{A}).$$

$\therefore B_1 = \emptyset$ which is a contradiction.

$\therefore B$ is connected.

Problem 2. If A and B are connected subsets of a metric space M and if $A \cap B \neq \emptyset$, prove that $A \cup B$ is connected.

Solution. Let $f: A \cup B \to \{0,1\}$ be a continuous function.

Since $A \cap B \neq \emptyset$, we can choose $x_0 \in A \cap B$.

Let $f(x_0) = 0$.

Since $f: A \cup B \to \{0,1\}$ is continuous $f|_A: A \to \{0,1\}$ is also continuous.

But A is connected.

Hence $f|_A$ is not onto.

$\therefore f(x) = 0$ for all $x \in A$ or $f(x) = 1$ for all $x \in A$.

But $f(x_0) = 0$ and $x_0 \in A$.

$$\therefore f(x) = 0 \text{ for all } x \in A.$$

Thus any continuous function $f: A \cup B \to \{0,1\}$ is not onto.

$$\therefore A \cup B \text{ is connected.}$$

12.3 CONNECTED SUBSETS OF R

Theorem 12.3 A subspace of R is connected iff it is an interval.

Proof. Let A be a connected subset of R.

Suppose A is not an interval.

Then there exist $a, b \in R$ such that $a < b < c$ and $a, c \in A$ but $b \notin A$.

Let $A_1 = (-\infty, b) \cap A$ and $A_2 = (b, \infty) \cap A$.

Since $(-\infty, b)$ and (b, ∞) are open in R, A_1 and A_2 are open sets in A.

Also $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$. Further $a \in A_1$ and $c \in A_2$.

Hence $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$.

Thus A is the union of two disjoint non-empty open sets A_1 and A_2.

Hence A is not connected which is a contradiction.

Hence A is an interval.

Conversely, let A be an interval. We claim that A is connected.

Suppose A is not connected. Let $A = A_1 \cup A_2$ where $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $A_1 \cap A_2 = \emptyset$ and A_1 and A_2 are closed sets in A.

Choose $x \in A_1$ and $z \in A_2$. Since $A_1 \cap A_2 = \emptyset$ we have $x \neq z$.

Without loss of generality we assume that $x < z$.

Now, since A is an interval we have $[x, z] \subseteq A$.

(i.e.) $[x, z] \subseteq A_1 \cup A_2$.

$\therefore$ Every element of $[x, z]$ is either in A_1 or in A_2.

Now, let $y = l.u.b.\{[x, z] \cap A_1\}$.

Clearly, $x \leq y \leq z$.

Hence $y \in A$.

Let $\varepsilon > 0$ be given. Then by the definition of *l.u.b* there exists $t \in [x, z] \cap A_1$ such that $y - \varepsilon < t \leq y$.

$$\therefore (y - \varepsilon, y + \varepsilon) \cap ([x, z] \cap A_1) \neq \emptyset.$$

$$\therefore y \in \overline{[x, z] \cap A_1}$$

$\therefore y \in [x, z] \cap A_1$ (since $[x, z] \cap A_1$ is closed in A).

Therefore, $y \in A_1$ ....(1)

Again by the definition of $y, y + \varepsilon \in A_2$ for all $\varepsilon > 0$ such that $y + \varepsilon \leq z$.

$$\therefore y \in \overline{A_2}$$

$\therefore y \in A_2$ (since A_2 is closed). ....(2)

$\therefore y \in A_1 \cap A_2$ [by (1) and (2)] which is a contradiction since $A_1 \cap A_2 = \emptyset$.

Hence A is connected.

Theorem 12.4 R is Connected.

Proof. $R = (-\infty, \infty)$ is an interval.

$\therefore R$ is connected.

Solved Problems

Problem 1. Give an example to show that a subspace of a connected metric space need not be connected.

Solution. We know that R is connected.

$A = [1,2] \cup [3,4]$ is a subspace of R which is not connected.

Problem 2. Prove or disprove if A and C are connected subsets of a metric space M and if $A \subseteq B \subseteq C$, then B is connected.

Solution. We disprove this statement by giving a counter example.

Let $A = [1,2]$; $B = [1,2] \cup [3,4]$; $C = R$.

Clearly $A \subset B \subset C$.

Here A and C are connected. But B is not connected.

NOTES

12.4 CONNECTEDNESS AND CONTINUITY

Theorem 12.5 Let M_1 be a connected metric space. Let M_2 be any metric space. Let $f: M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.

$(i.e.)$ *Any continuous image of a connected set is connected.*

Proof. Let $f(M_1) = A$ so that f is a function from M_1 onto A.

We claim that A is connected.

Suppose A is not connected. Then there exists a proper non-empty subset B of A which is both open and closed in A.

$\therefore f^{-1}(B)$ is a proper non-empty subset of M_1 which is both open and closed in M_1. Hence M_1 is not connected which is a contradiction.

Hence A is connected.

Theorem 12.6 Let f be a real valued continuous function defined on an interval I. Then f takes every value between any between any two values it assumes. (*This if known as* ***the intermediate value theorem***).

Proof. Let $a, b \in I$ and let $f(a) \neq f(b)$.

Without loss of generality we assume that $f(a) < f(b)$.

Let c be such that $f(a) < c < f(b)$.

The interval I is a connected subset of R. Therefore, $f(I)$ is a connected subset of R. (by theorem 12.5)

$\therefore f(I)$ is an interval. (by theorem 12.3)

Also $f(a), f(b) \in f(I)$. Hence $[f(a), f(b)] \subseteq f(I)$.

$\therefore c \in f(I)$ [since $f(a) < c < f(b)$]

$\therefore c = f(x)$ for some $x \in I$.

Solved Problems

NOTES

Problem 1. Prove that if f is a non-constant real valued continuous function on R then the range of f is uncountable.

Solution. We know that R is connected.

Since f is a continuous function on R, $f(R)$ is a connected subset of R.

$\therefore f(R)$ is an interval in R.

Also, since f is a non-constant function the interval. $f(R)$ contains more than one point.

$\therefore f(R)$ is uncountable. (i.e.) the range of f is uncountable.

Check your progress

1) Define connected set and disconnected set.
2) Any continuous image of a connected set is? connected.
3) What about R? Connected or disconnected.

12.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let (M, d) be a metric space. M is said to be *connected* if M cannot be represented as the union of two disjoint non-empty open sets.

If M is not connected it is said to be *disconnected*.

2. Any continuous image of a connected set is connected.
3. R is Connected.

12.6 SUMMARY

- Let (M, d) be a metric space. M is said to be ***connected*** if M cannot be represented as the union of two disjoint non-empty open sets.

  If M is not connected it is said to be ***disconnected***.

- Let $M = [1,2] \cup [3,4]$ with usual metric. Then M is disconnected.
- Any discrete metric space M with more than one point is disconnected.
- Let (M, d) be a metric space. Then the following are equivalent.

  v. M is connected.

  vi. M cannot be written as the union of two disjoint non-empty closed sets.

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vii. M cannot be written as the union of two non-empty sets A and B such that $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

viii. M and $\emptyset$ are the only sets which are both open and closed in M.

- A metric space M is connected iff there does not exist a continuous function f from M onto the discrete metric space $\{0,1\}$.
- *M is connected iff every continuous function $f: M \to \{0,1\}$ is not onto.*
- A subspace of R is connected iff it is an interval.
- R is Connected
- Let M_1 be a connected metric space. Let M_2 be any metric space. Let $f: M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.
- Let f be a real valued continuous function defined on an interval I. Then f takes every value between any between any two values it assumes. (*This if known as* ***the intermediate value theorem***).

12.7 KEYWORDS

Connected: Let (M, d) be a metric space. M is said to be ***connected*** if M cannot be represented as the union of two disjoint non-empty open sets.

Disconnected: If M is not connected it is said to be ***disconnected***.

12.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. Let $\{A_\alpha\}$ be a family of connected subsets of a metric space M such that $\cap A_\alpha \neq \emptyset$. Then prove that $A = \cup A_\alpha$ is a connected subset of M.
2. Prove that the set of all components of a metric space M forms a partition of M.
3. Let $A_1, A_2, \dots, A_n, \dots$ be connected subsets of a metric space M each of which intersects its successor. Prove that $\cup_{n=1}^{\infty} A_n$ is connected.
4. Prove that any connected subset of R containing more than one point is uncountable.
5. If M is a metric space and $x \in M$ then $\{x\}$ is a connected subset of M.

12.9 FURTHER READINGS

NOTES

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4) M.K. Singhal & Asha Rani Singhal, A First Course in Real Analysis, R. Chand & Co. June 1997 Edition.

5) Shanthi Narayan, A Couse of Mathematical Analysis, S. Chand & Co., 1995

NOTES

UNIT 13 COMPACTNESS

Structure

13.0 INTRODUCTION

We have seen that the concept of completeness is the abstraction of a property of the real number system. The concept of compactness is also an abstraction of an important property possessed by subsets of $\boldsymbol{R}$ which are closed and bounded. This property is known as Heine Borel theorem which states that if $I \subseteq \boldsymbol{R}$ is a closed interval, any family of open intervals in $\boldsymbol{R}$ whose union contains I has a finite subfamily whose union contains I.We now introduce the class of *compact metric spaces* in which the conclusion of Heine Borel theorem is valid.

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is Complete metric space
- Discuss Compact Subset of R

13.2 COMPLETE METRIC SPACE

13.2.1 Definition and Examples

Definition: Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an *open cover* for M if $\cup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a *subcover.*

A metric space M is said to be *compact* if every open cover for M has finite subcover.

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(i.e.) for each family of open sets $\{G_\alpha\}$ such that $\cup G_\alpha = M$, there exist a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $\cup_{i=1}^{n} G_{\alpha_i} = M$.

Example 1. R with usual metric is not compact.

Proof. Consider the family of open intervals $\{(-n,\ n)/\ n \in \mathrm{N}\}$.

This is a family of open sets in R.

Clearly $\cup_{i=1}^{n}(-n,\ n) = R$.

$\therefore$ $\{(-n,\ n)/\ n \in \mathrm{N}\}$ is an open cover for R and this open cover has no finite subcover.

$\therefore$ R is not compact.

Example 2. $(0, 1)$ with usual metric is not compact.

Proof. Consider the family of open intervals $\{(1/n, 1)/n = 2, 3, \dots\}$

Clearly $\cup_{n=2}^{\infty}(\frac{1}{n}, 1) = (0, 1)$.

$\therefore$ $\{(1/n, 1)/n = 2, 3, \dots\}$ is an open cover for (0, 1) and this open cover has no finite subcover.

Hence $(0, 1)$ is not compact.

Example 3. $[0, \infty)$ with usual metric is not compact.

Proof. Consider the family of intervals $\{[0, n)/n \in N\}$.

$[0, n)$ is open in $[0, \infty)$ for each $n \in N$.

Also $\cup_{n=1}^{\infty} (0, n) = [0, \infty)$.

$\therefore$ $\{[0, n)/n \in N\}$ is an open cover for $[0, \infty)$ and this open cover has no finite subcover.

Hence $[0, \infty)$ is not compact.

Example 4. Let M be an infinite set with discrete metric. Then M is not compact.

Proof. Let $x \in M$. Since M is a discrete metric space $\{x\}$ is open in M.

Also $\cup_{x \in M}\{x\} = M$.

Hence $\{\{x\}/x \in M\}$ is an open cover for M and since M is infinite, this open cover has no finite subcover.

Hence M is not compact.

NOTES

Example 5. Any closed interval $[a, b]$ with usual metric is compact.

Theorem 13.1 Let M be a metric space. Let $A \subseteq M$. A is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\cup G_\alpha \supseteq A$ there exists a subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = A$.

Proof. Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a family of open sets in M such that $\cup G_\alpha \supseteq A$.

Then $(\cup G_\alpha) \cap A = A$.

$\therefore \cup (G_\alpha \cap A) = A$.

Also $G_\alpha \cap A$ is open in A.

$\therefore$ The family $\{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact this open cover has a finite subcover, say, $G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \dots, G_{\alpha_n} \cap A$.

$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Conversely, let $\{H_\alpha\}$ be an open cover for A.

$\therefore$ Each H_α is open in A.

$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M.

Now, $\cup H_\alpha = A$.

$\therefore \cup (G_\alpha \cap A) = A$.

$\therefore (\cup G_\alpha) \cap A = A$.

$\therefore \cup G_\alpha \supseteq A$

Hence by hypothesis there exists a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$.

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$\therefore \bigcup_{i=1}^{n} H_{\alpha_i} = A.$

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}\}$ is a finite subcover of the open cover $\{H_\alpha\}$.

$\therefore A$ is compact.

Theorem 13.2 Any compact subset A of a metric space M is bounded.

Proof. Let $x_0 \in A$.

Consider $\{B(x_0, n)/n \in N\}$.

Clearly $\bigcup_{n=1}^{\infty} B(x_0, n) = M$.

$\therefore \bigcup_{n=1}^{\infty} B(x_0, n) \supseteq A$.

Since A is compact there exists a finite subfamily say, $B(x_0, n_1), B(x_0, n_2), \dots, B(x_0, n_k)$ such that $\bigcup_{i=1}^{k} B(x_0, n_i) \supseteq A$.

Let $n_0 = \max\{n_1, n_2, \dots, n_k\}$.

Then $\bigcup_{i=1}^{k} B(x_0, n_i) = B(x_0, n_0)$.

$\therefore B(x_0, n_0) \supseteq A$.

We know that $B(x_0, n_0)$ is a bounded set and a subset of a bounded set is bounded. Hence A is bounded.

Note:

The converse of the above theorem is not true.

For example, $(0, 1)$ is a bounded subset of R. But it is not compact.

Theorem 13.3 Any compact subset A of a metric space(M, d) is closed.

Proof. To prove that A is closed we shall prove that A^c is open.

Let $y \in A^c$ and let $x \in A$. Then $x \neq y$.

$\therefore d(x, y) = r_x > 0$.

It can be easily verified that $B\left(x, \frac{1}{2} r_x\right) \cap B\left(y, \frac{1}{2} r_x\right) = \emptyset$.

Now consider the collection $\{B\left(x, \frac{1}{2} r_x\right)/x \in A\}$.

Clearly $\bigcup_{x \in A} B\left(x, \frac{1}{2} r_x\right) \supseteq A$.

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Since A is compact there exists a finite number of such open balls say, $B\left(x_1, \frac{1}{2}r_{x_1}\right), \dots, B\left(x_n, \frac{1}{2}r_{x_n}\right)$ such that

$\bigcup_{i=1}^{n} B\left(x_i, \frac{1}{2}r_{x_i}\right) \supseteq A$.------------------->(1)

Now, let $V_y = \bigcap_{i=1}^{n} B\left(y, \frac{1}{2}r_{x_i}\right)$

Clearly V_y is an open set containing y.

Since $B\left(y, \frac{1}{2}r_y\right) \cap B\left(x, \frac{1}{2}r_x\right) = \emptyset$, we have $V_y \cap B\left(x, \frac{1}{2}r_{x_i}\right) = \emptyset$ for each $i = 1, 2, \dots, n$.

$\therefore V_y \cap [\bigcup_{i=1}^{n} B\left(x, \frac{1}{2}r_{x_i}\right)] = \emptyset$.

$\therefore\ V_y \cap A = \emptyset$ [by (1)].

$\therefore V_y \subseteq A^c$.

$\therefore \bigcup_{y \in A^c} V_y = A^c$ and each V_y is open.

$\therefore\ A^c$ is open. Hence A is closed.

Note 1. The converse of the above theorem is not true.

For example, $[0, 1)$ is a closed subset of R. But it is not compact.

Note 2. It follows from the above two theorems that any compact subset of a metric space is closed and bounded.

Theorem 13. 3 A closed subspace of a compact metric space is compact.

Proof. Let M be a compact metric space. Let A be a non-empty closed subset of M.

We claim that A is compact.

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M such that $\bigcup_{\alpha \in I} G_\alpha \supseteq A$.

$\therefore\ A^c \cup (\bigcup_{\alpha \in I} G_\alpha) = M$.

Also A^c is open, since A is closed.

$\therefore\ \{G_\alpha / \alpha \in I\} \cup \{A^c\}$ is an open cover for M.

Since M is compact it has a finite subcover say $G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}, A^c$.

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$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cup A^c = M.$

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$

$\therefore A$ is compact.

13.3 COMPACT SUBSETS OF R

We have already proved that compact subset of a metric space is closed and bounded.

However the converse is true.

For example, consider an infinite discrete metric space (M, d).

Let A be an infinite subset of M.

Then A is bounded since $d(x, y) \leq 1$ for all $x, y \in A$.

Also A is closed since any subset of a discrete metric space is closed.

Hence A is closed and bounded.

However A is not compact.

In this section we shall prove that for R with usual metric the converse is also true.

Theorem 13. [Heine Borel Theorem]

Any closed interval $[a, b]$ is a compact subset of R.

Proof.

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in R such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a, b]$.

Let $S = \{x / x \in [a, b]$ } and $[a, x]$ can be covered by a finite number of G_α's.

Clearly $a \in S$ and hence $S \neq \emptyset$.

Also S is bounded above by b.

Let c denote the l. u. b. of S.

Clearly $c \in [a, b]$

$\therefore c \in G_{\alpha_i}$ for some $\alpha_i \in I$.

Since G_{α_i} is open, there exists $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq G_{\alpha_i}$.

Choose $x_1 \in [a, b]$ such that $x_1 < c$ and $[x_1, c] \subseteq G_{\alpha_i}$.

Now, since $x_1 < c$, [a, x_1] can be covered by a finite number of G_α's.

These finite number of G_α's together with G_{α_i} covers $[a, c]$.

∴ By definition of S, $c \in S$.

Now, we claim that $c = b$.

Suppose $c \neq b$.

Then choose $x_2 \in [a, b]$ such that $x_2 > c$ and $[c, x_2] \subseteq G_{\alpha_i}$.

As before, $[a, x_2]$ can be covered by a finite number of G_α's.

Hence $x_2 \in S$.

But $x_2 > c$ which is a contradiction, since c is the l. u. b. of S.

∴ $c = b$.

∴ $[a, b]$ can be covered by a finite number of G_α's.

∴ $[a, b]$ is a compact subset of R.

Theorem 13. A subset A of R is compact iff A is closed and bounded.

Proof. If A is compact then A is closed and bounded.

Conversely, let A be subset of R which is closed and bounded.

Since A is bounded we can find a closed interval $[a, b]$ such that $A \subseteq [a, b]$.

Since A is closed in R, A is closed interval $[a, b]$ also.

Thus A is a closed subset of the compact space $[a, b]$.

Hence A is compact.

Definition. A family $\mathcal{T}$ of subsets of a set M is said to have the finite intersection property if any finite members of $\mathcal{T}$ have non-empty intersection.

Example. In R the family of closed intervals $\mathcal{T} = \{[-n, n]/n \in N\}$ has finite intersection property.

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Theorem A metric space M is compact iff any family of closed sets with finite intersection property has non-empty intersection.

Proof. Suppose M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

We claim that $\cap A_\alpha \neq \emptyset$.

Suppose $\cap A_\alpha = \emptyset$ then $(\cap A_\alpha)^c = \emptyset^c$.

$\therefore \cup {A^c}_\alpha = M$.

Also, since each A_α is closed, ${A^c}_\alpha$ is open.

$\therefore \{{A^c}_\alpha\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say, ${A^c}_1, {A^c}_2, \ldots, {A^c}_n$.

$\therefore \bigcup_{i=1}^{n} {A^c}_i = M$.

$\therefore (\bigcap_{i=1}^{n} A_i)^c = M$.

$\therefore \bigcap_{i=1}^{n} A_i = \emptyset$ which is a contradiction to the finite intersection property.

$\therefore \cap A_\alpha \neq \emptyset$.

Conversely, suppose that each family of closed sets in M with finite intersection property has non-empty intersection.

To prove that M is compact, let $\{G_\alpha / \alpha \in I\}$ be an open cover for M.

$\therefore \bigcup_{\alpha \in I} G_\alpha = M$.

$\therefore (\bigcup_{\alpha \in I} G_\alpha)^c = M^c$.

$\therefore \bigcap_{\alpha \in I} {G_\alpha}^c = \emptyset$.

Since G_α is open, ${G_\alpha}^c$ is closed for each α.

$\therefore \mathcal{T} = \{{G_\alpha}^c / \alpha \in I\}$ is a family of closed sets whose intersection is empty.

Hence by hypothesis this family of closed sets does not have the finite intersection property.

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Hence there exists a finite sub-collection of $\mathcal{T}$ say, $\{G_1{}^c, G_2{}^c, \dots, G_n{}^c\}$ such that $\bigcap_{i=1}^{n} G_i{}^c = \emptyset$.

$\therefore (\bigcup_{i=1}^{n} G_i)^c = \emptyset$.

$\therefore \bigcup_{i=1}^{n} G_i = M$.

$\therefore \{G_1, G_2, \dots, G_n\}$ is a finite subcover of the given open cover.

Hence M is compact.

Definition.

A metric space M is said to be totally bounded if for every $\varepsilon > 0$ there exists a finite number of elements $x_1, x_2, \dots, x_n \in M$

such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \dots \cup B(x_n, \varepsilon) = M$.

A non-empty subset A of a metric space M is said to be totally bounded if the subspace A is a totally bounded metric space.

Theorem. Any compact metric space is totally bounded.

Proof. Let M be a compact metric space.

Then $\{B(x, \varepsilon) / x \in M\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say,

$B(x_1, \varepsilon), B(x_2, \varepsilon), \dots, B(x_n, \varepsilon)$.

$\therefore M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \dots \cup B(x_n, \varepsilon)$.

$\therefore M$ is totally bounded.

Theorem Let A be a subset of a metric space M. If A is totally bounded then A is bounded.

Proof. Let A be a totally bounded subset of M. Let $\varepsilon > 0$ be given.

Then there exists a finite number of points $x_1, x_2, \dots, x_n \in A$, such that

$B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \dots \cup B(x_n, \varepsilon) = A$, where $B(x_i, \varepsilon)$ is an open ball in A.

Further we know that an open ball is a bounded set.

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Thus A is the union of finite number of bounded sets and hence A is bounded.

Note. The converse of the above theorem is not true.

For, let M be an infinite set with discrete metric.

Clearly M is bounded.

Now, $B\left(x, \frac{1}{2}\right) = \{x\}$.

Since M is infinite, M cannot be written as the union of a finite number of open balls $B\left(x, \frac{1}{2}\right)$.

$\therefore$ N is not totally bounded.

Definition. Let (x_n) be sequence in a metric space M. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be an increasing sequence of positive integers. Then (x_{n_k}) is called a subsequence of (x_n).

Theorem. A metric space (M, d) is totally bounded iff every sequence in M has a Cauchy subsequence.

Proof. Suppose every sequence in M has a Cauchy subsequence.

We claim that M is totally bounded.

Let $\varepsilon > 0$ be given. Choose $x_1 \in M$.

If$B(x_1, \varepsilon) = M$ then obviously M is totally bounded.

If $B(x_1, \varepsilon) \neq M$, choose $x_2 \in M - B(x_1, \varepsilon)$ so that $d(x_1, x_2) \geq \varepsilon$.

Now , if $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) = M$ the proof is complete.

If not choose $x_3 = M - [B(x_1, \varepsilon) \cup B(x_2, \varepsilon)]$ and so on.

Suppose this process does not stop at a finite stage.

Then we obtain a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $d(x_n, x_m) \geq \varepsilon$ if $n \neq m$.

Clearly this sequence (x_n) can not have a Cauchy sequence which is a contradiction.

Hence the above process stops at a finite stage and we get a finite set of points $\{x_1, x_2, \ldots, x_n\}$ such that $M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots . \cup B(x_n, \varepsilon)$.

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$\therefore$ M is totally bounded.

Conversely suppose M is totally bounded.

Let $S_1 = \{x_{i_1}, x_{i_2}, \dots, x_{i_n}, \dots\}$ be a sequence in M.

If one term of the sequence is infinitely repeated then S_1 contains a constant subsequence which is obviously a Cauchy subsequence.

Hence we assume that no term of S_1 is infinitely repeated so that the range of S_1 is infinite.

Now, since M is totally bounded M can be covered by a finite number of open balls of radius $1/2$.

Hence at least one of these balls must contain an infinite number of terms of the sequence S_1.

$\therefore$ S_1 contains a subsequence $S_2 = (x_{2_1}, x_{2_2}, \dots, x_{2_n}, \dots)$ all terms of which lie within an open ball of radius $1/2$.

Similarly S_2 contains a subsequence $S_3 = (x_{3_1}, x_{3_2}, \dots, x_{3_n}, \dots)$ all terms of which lie within an open ball of radius $1/3$.

We repeat this process of forming successive subsequences and finally we take the diagonal sequence $S = (x_{1_1}, x_{2_2}, \dots, x_{n_n}, \dots)$.

We claim that S is a Cauchy subsequence of S_1.

If $m > n$ both x_{m_m} and x_{n_n} lie within an open ball of radius $1/n$.

$\therefore d(x_{m_m}, x_{n_n}) < 2/n$.

Hence $d(x_{m_m}, x_{n_n}) < \varepsilon$ if $n, m > 2/\varepsilon$.

This shows that S is a Cauchy subsequence of S_1.

Thus every sequence in M contains a Cauchy subsequence.

Corollary. A non-empty subset of a totally bounded set id totally bounded.

Proof. Let A be a totally bounded subset of a metric space M.

Let B be a non-empty subset of A.

Let (x_n) be a sequence in B.

$\therefore$ (x_n) is a sequence in A.

NOTES

Since A is a totally bounded (x_n) has a Cauchy subsequence.

Thus every sequence in B has a Cauchy subsequence.

$\therefore B$ is totally bounded.

Definition

A metric space M is said to be sequentially compact if every sequence in M has a convergent sun-sequence.

Theorem

Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence (x_{n_k}) converging to x, then (x_n) converges to x.

Proof. Let $\varepsilon > 0$ be given. Since (x_n) is a Cauchy sequence, there exists a positive integer m_1 such that $d(x_n, x_m) < \frac{1}{2}\varepsilon$ for all $n, m \geq m_1$ ----------------------------> (1)

Also, since $(x_{n_k}) \to x$, there exists a positive integer m_2 such that $d(x_{n_k}, x) < \frac{1}{2}\varepsilon$ for all $n_k \geq m_2$ ---------------------> (2)

Let $m_0 = \max\{m_1, m_2\}$ and fix $n_k \geq m_0$.

Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ for all } n \geq m_0 \text{ by (1) and (2)}$$

$$= \varepsilon \text{ for all } n \geq m_0.$$

Hence $(x_n) \to x$.

Theorem In a metric space M the following are equivalent.

i. M is compact.
ii. Any infinite subset of M has a limit point.
iii. M is sequentially compact.
iv. M is totally bounded and complete.

Proof.

$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ **.** Let A be an infinite subset of M.

Suppose A has no limit point in M.

Let $x \in M$.

NOTES

Since x is not a limit point of A there exists an open ball $B(x,\ r_x)$ such that $B(x,\ r_x) \cap (A - \{x\}) = \emptyset$.

$$\therefore\ B(x,\ r_x) \cap A = \begin{cases} \{x\} & if\ \ x \in A \\ \emptyset & if\ \ x \notin A \end{cases}$$

Now, $\{B(x,\ r_x)/x \in M\}$ is open cover for M.

Also each $B(x,\ r_x)$covers at most one point of the infinite set A.

Hence this open cover can not have a finite sub cover which is a contradiction to (i). Hence A has at least one limit point.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$ **.** Let (x_n) be a sequence in M.

If one term of the sequence is infinitely repeated, then (x_n) contains a constant subsequence which is convergent.

Otherwise (x_n) has an infinite number of terms.

By hypothesis this infinite set has a limit point, say x.

We know that for any $r > 0$ the open ball $B(x, r)$ contains infinite number of terms of the sequence (x_n) .

Now, choose a positive integer n, such that $x_{n_1} \in B(x, 1)$.

Then choose $n_2 > n_1$ such that $x_{n_2} \in B(x, {}^1\!/_2)$.

In general for each positive integer k choose n_k such that $n_k > n_{k-1}$ and

$$x_{n_k} \in B(x, {}^1\!/_k).$$

Clearly (x_{n_k}) is a subsequence of (x_n) .

Also $d\big(x_{n_k}, x\big) < {}^1\!/_k$.

$$\therefore (x_{n_k}) \to x.$$

Thus (x_{n_k}) is a convergent subsequence of (x_n) .

Hence M is sequentially compact.

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{iv})$ **.** By hypothesis every sequence in M has a convergent subsequence.

NOTES

But every convergent sequence is a Cauchy sequence.

Thus every sequence in M has a Cauchy subsequence.

$\therefore$ By the theorem, M is totally bounded.

Now, we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

By hypothesis (x_n) contains a convergent subsequence $\left(x_{n_k}\right)$.

Let $\left(x_{n_k}\right) \to x$. (say)

Then by previous theorem, $(x_n) \to x$.

$\therefore$ M is complete.

$(\boldsymbol{iv}) \Rightarrow (\boldsymbol{i})$. Suppose M is not compact.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Let $r_n = \frac{1}{2^n}$.

Since M is totally bounded, M can be covered by a finite number of open balls of radius r_1.

Since M can not be covered by a finite number of G_α's at least one of these open balls, say $B(x_1, r_1)$ cannot be covered by a finite number of G_α's.

Now, $B(x_1, r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1, r_1)$ such that $B(x_2, r_2)$ cannot be covered by a finite number of G_α's.

Proceeding like this we obtain a sequence (x_n) in M such that $B(x_n, r_n)$ cannot be covered by a finite number of G_α's and $x_{n+1} \in B(x_n, r_n)$ for all n.

Now,

$$d\left(x_n, x_{n+p}\right) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d\left(x_{n+p-1}, x_{n+p}\right)$$

$$< r_n + r_{n+1} + \cdots + r_{n+p-1}$$

$$= \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}}$$

$$= \frac{1}{2^{n-1}}\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^p}\right) < \frac{1}{2^{n-1}}.$$

$$= \frac{1}{2^{n-1}}\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^p}\right) < \frac{1}{2^{n-1}}.$$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Now, $x \in G_\alpha$ for some α.

Since G_α is open we can find $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq G_\alpha$------>(1)

We have $(x_n) \to x$ and $(r_n) = \left(\frac{1}{2^n}\right) \to 0$.

Hence we can find a positive integer n_1 such that $d(x_n, x) < \frac{1}{2}\varepsilon$ and $r_n < \frac{1}{2}\varepsilon$ for all $n \geq n_1$.

Now, fix $n \geq n_1$.

We claim that $B(x_n, r_n) \subseteq B(x,\varepsilon)$.

Let $y \in B(x_n, r_n)$

$\therefore$ $d(y, x_n) < r_n < \frac{1}{2}\varepsilon$, since $n \geq n_1$.

Now, $d(y,x) \leq d(y, x_n) + d(x_n, x)$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$\therefore y \in B(x,\varepsilon)$.

$\therefore$ $B(x_n, r_n) \subseteq B(x,\varepsilon) \subseteq G_\alpha$, by (1)

Thus $B(x_n, r_n)$ is covered by the single set G_α which is a contradiction since $B(x_n, r_n)$ cannot be covered by a finite number of G_α's.

Hence M is compact.

Theorem R with usual metric is complete.

Proof. Let (x_n) be a Cauchy sequence in R.

Then (x_n) is a bounded sequence and hence is contained in a closed interval $[a,b]$.

NOTES

Now, $[a, b]$ is compact and hence is complete.

Hence (x_n) converges to some point $x \in [a, b]$.

Thus every Cauchy sequence (x_n) in R converges to some point x in R and hence R is complete.

Solved Problems:

1. Give an example of a closed and bounded subset of l_2 which is not compact.

Solution: Consider $0 = (0,0, \dots) \in l_2$.
Consider the closed ball $B[0,1]$.
Clearly, $B[0,1]$ is bounded.
Also, $B[0,1]$ is a closed set.
We claim that $B[0,1]$ is not compact.
Consider $e_1 = (1,0,0, \dots); e_2 = (0,1,0, \dots); \dots e_n = (0,0,0, \dots ,1,0, \dots)$.
Now, $d(0, e_n) = 1$ and hence $e_n \in B[0,1]$ for all n.
Thus (e_n) is a sequence in $B[0,1]$.
Also $d(e_n, e_m) = \sqrt{2}$ if $n \neq m$.
Hence the sequence (e_n) does not contain a Cauchy subsequence.
$\therefore B[0,1]$ is not totally bounded.
$\therefore B[0,1]$ is not compact.

Problem 2:

Prove that any totally bounded metric space is separable.

Solution: Let M be a totally bounded metric space.
For each natural number n let $A_n = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}\}$ be a subset of M such that $\bigcup_{i=1}^{k} B\left(x_n, \frac{1}{n}\right) = M$. ------------------>(1)
Let $A = \bigcup_{n=1}^{\infty} A_n$.
Since each A_n is finite, A is a countable subset of M.

We claim that A is dense in M.

Let $B(x, \varepsilon)$ be any open ball.

Choose a natural number n such that $1/n < \varepsilon$.

Now, $x \in B(x_{n_i}, \frac{1}{n})$ for some i, by (1)

$\therefore B(x_{n_i}, x) < {}^{1}/_{n} < \varepsilon$.

$\therefore \left(x_{n_i}\right) \in B(x, \varepsilon)$

$\therefore \; B(x, \varepsilon) \cap A \neq \emptyset$.

Thus every open ball in M has non-empty intersection with A.

Hence A is dense in M.

Thus A is a countable dense subset of M.

Hence M is separable.

Problem 3. Prove that any bounded sequence in R has a convergent subsequence.

Solution. Let (x_n) be a bounded sequence in R.

Then there exists a closed interval $[a, b]$ such that $(x_n) \in [a, b]$ for all n.

Thus (x_n) is a sequence in the compact metric space $[a, b]$.

Hence by the above theorem, (x_n) has a convergent sub-sequence.

Problem 4. Prove that the closure of a totally bounded set is totally bounded.

Solution. Let A be a totally bounded subset of a metric space M.

We claim that $\bar{A}$ is a totally bounded.

We shall show that every sequence in $\bar{A}$ contains a Cauchy subsequence.

Let (x_n) be a sequence in $\bar{A}$.

Let $\varepsilon > 0$ be given.

Then since $x_n \in \bar{A}, B\left(x_n, \frac{1}{3}\varepsilon\right) \cap A \neq \emptyset$.

Choose $y_n \in B\left(x_n, \frac{1}{3}\varepsilon\right) \cap A$.

$\therefore \; d(y_n, x_n) < \frac{1}{3}\varepsilon$.---------------->(1)

Now, (y_n) is a sequence in A. Since A is totally bounded (y_n) contains a Cauchy sequence say (y_{n_k}).

Hence there exists a natural number m such that

$$d\left(y_{n_i}, y_{n_j}\right) < \frac{1}{3}\varepsilon \quad \text{for all } n_j, \; n_i \geq m \text{ --------->(2)}$$

NOTES

$$\therefore\ d\left(x_{n_i}, x_{n_j}\right) \leq d(x_{n_i}, y_{n_i}) + d\left(y_{n_i}, y_{n_j}\right) + d\left(y_{n_j}, x_{n_j}\right)$$

$$< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon \text{ for all } n_j,\ n_i \geq m \text{, by (1)and (2)}$$

Hence (x_{n_k}) is a Cauchy subsequence of (x_n) .

$\therefore$ $\bar{A}$ is totally bounded.

Problem 5. Let A be a totally bounded subset of R. Prove that $\bar{A}$ is compact.

Solution. Since A is totally bounded, $\bar{A}$ is also totally bounded.

Also, since $\bar{A}$ is a closed subset of R and R is complete $\bar{A}$ is complete.

Hence $\bar{A}$ is totally bounded and complete.

$\therefore$ $\bar{A}$ is compact.

Theorem Let f be a continuous mapping from a compact metric space M_1 to any metric space M_2. Then $f(M_1)$ is compact.

Proof. Without loss of generality we assume that $f(M_1) = M_2$.

Let $\{G_\alpha\}$ be a family of open sets in M_2 such that $\bigcup G_\alpha = M_2$.

$\therefore \bigcup G_\alpha = f(M_1)$.

$\therefore f^{-1}(\bigcup G_\alpha) = M_1$.

$\therefore \bigcup f^{-1}(G_\alpha) = M_1$.

Also since f is continuous $f^{-1}(G_\alpha)$ is open in M_1 for each α.

$\therefore \{f^{-1}(G_\alpha)\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover, say,

$f^{-1}\left(G_{\alpha_1}\right), f^{-1}\left(G_{\alpha_2}\right), \ldots, f^{-1}\left(G_{\alpha_n}\right)$.

$\therefore$ $f^{-1}\left(G_{\alpha_1}\right) \cup f^{-1}\left(G_{\alpha_2}\right) \cup \ldots \cup f^{-1}\left(G_{\alpha_n}\right) = M_1$.

$\therefore$ $f^{-1}\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) = M_1$.

$\therefore$ $\bigcup_{i=1}^{n} G_{\alpha_i} = f(M_1) = M_2$.

$\therefore$ $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ is a cover for M_2.

Thus the given open cover $\{G_\alpha\}$ for M_2 has a finite subcover.

$\therefore$ M_2 is compact.

Corollary 1. Let f be a continuous map from a compact metric space M_1 into any metric M_2. Then $f(M_1)$ is closed and bounded.

Proof. $f(M_1)$ is compact and hence is closed and bounded.

Corollary 2. Any continuous real valued function f defined on a compact metric space is bounded and attains its bounds.

Proof. Let M be a compact metric space.

Let $f: M \to R$ be a continuous real valued function.

Then $f(M)$ is a compact subset of R.

$\therefore$ $f(M)$ is a closed and bounded subset of R.

Since $f(M)$ is bounded f is a bounded function.

Now, let $a = l.\ u.\ b.\ of\ \ f(M)$ and $b = g.\ l.\ b.\ of\ \ f(M)$.

By definition of l. u. b and g. l. b $a, b \in \overline{f(M)}$

But $f(M)$ is closed . Hence $f(M) = \overline{f(M)}$.

$\therefore$ $a, b \in f(M)$.

$\therefore$ There exists $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

Hence f attains its bounds.

Note.

1. Corollary (2) is not true if M is not compact.
2. The function $f: (0,1) \to R$ defined by $f(x) = 1/x$ is continuous but not bounded.
3. The function $g: (0,1) \to R$ defined by $g(x) = x$ is bounded having $l.u.b. = 1$ and $g.l.b. = 0$. However this function never attains these bounds at any point in $(0,1)$.

Theorem Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof. Let $\varepsilon > 0$ be given. Let $x \in M_1$.

Since f is continuous at x there exists $\delta_x > 0$ such that

NOTES

$$d_1(y,x) < \delta_x \Rightarrow d_2(f(y), f(x)) < {}^1\!/_2\,\varepsilon \text{ ---------->(1)}$$

Now, the family of open balls $\{B(x, \frac{1}{2}\delta_x)/x \in M_1\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite sub cover say

$B\left(x_1, \frac{1}{2}\delta_{x_1}\right), \dots, B\left(x_n, \frac{1}{2}\delta_{x_n}\right)$.

Let $\delta = \min\{\frac{1}{2}\delta_{x_1}, \dots, \frac{1}{2}\delta_{x_n}\}$.

We claim that $d_1(p,q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$.

Let $p \in B\left(x_i, \frac{1}{2}\delta_{x_i}\right)$ for some i where $1 \le i \le n$.

$$\therefore d_1(p, x_i) < \frac{1}{2}\delta_{x_i}$$

$\therefore d_2(f(p), f(x_i)) < {}^1\!/_2\,\varepsilon$, by (1) -------------->(2)

Now, $d_1(q, x_i) \le d_1(q, p) + d_1(p, x_i)$

$\le \delta + \frac{1}{2}\delta_{x_i}$

$\le \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i} = \delta_{x_i}$.

Thus $d_1(q, x_i) < \delta_{x_i}$.

$\therefore d_2(f(q), f(x_i)) < {}^1\!/_2\,\varepsilon$, by (1) ------------>(3)

Now, $d_2(f(p), f(q)) \le d_2(f(p), f(x_i)) + d_2(f(x_i), f(q))$

$< {}^1\!/_2\,\varepsilon + {}^1\!/_2\,\varepsilon = \varepsilon$. (by (2) and (3))

Thus $d_1(p,q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$.

This proves that f is uniformly continuous on M_1.

Note. The above theorem is not true if M_1 is not compact.

We have seen that if f is a continuous bijection then f^{-1} need not be continuous. Now we shall prove that if f is a continuous bijection defined on a compact metric space, then f^{-1} is also continuous.

NOTES

Theorem Let f be a $1-1$ continuous function from a compact metric space M_1 onto any metric space M_2. Then f^{-1} is continuous on M_2. Hence f is a homeomorphism from M_1 onto M_2.

Proof. We shall show that f^{-1} is continuous by proving that F is a closed set in M_1.

$\Rightarrow$ $(f^{-1})^{-1}(F) = f(F)$ is a closed set in M_2.

Let F be a closed set in M_1.

Since M_1 is compact F is compact.

Since f is continuous $f(F)$ is a compact subset of M_2.

$\therefore$ $f(F)$ is a closed subset of M_2.

$\therefore$ f^{-1} is continuous on M_2.

Solved Problems.

1. Prove that the range of a continuous real valued function f on a compact connected metric space M must be either a single point or a closed and bounded interval.

Solution. Let $f: M \to R$ be a continuous function.

Case(i). Suppose f is a constant function.

Then the range of f is a single point.

Case(ii). Suppose f is not a constant function.

Then the range of f contains more than one point.

since M is connected $f(M)$ is a connected subset of R.

$\therefore$ $f(M)$ is an interval in R.

Also, since M is compact and f is continuous $f(M)$ is a compact subset of M.

$\therefore$ $f(M)$ is a closed and bounded subset of R.

Thus $f(M)$ is a closed and bounded interval of R.

Problem 2. Prove that any continuous function $f: [a, b] \to R$ is not onto.

Solution. Suppose f is onto. Then $f([a, b]) = R$.

NOTES

Now, since $[a, b]$ is compact and f is continuous, $f([a, b]) = R$ is compact which is a contradiction.

$\therefore f$ is not onto.

Check your progress

1. Define open cover.
2. Define subcover.
3. Define compact metric space.
4. State Heine Borel theorem.

13.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an *open cover* for M if $\bigcup G_\alpha = M$.
2. A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a *subcover.*
3. A metric space M is said to be *compact* if every open cover for M has finite subcover.
4. Any closed interval $[a, b]$ is a compact subset of R.

13.5 SUMMARY

- Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an ***open cover*** for M if $\bigcup G_\alpha = M$.
- A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a ***subcover.***
- A metric space M is said to be ***compact*** if every open cover for M has finite subcover.
- R with usual metric is not compact.
- $(0, 1)$ with usual metric is not compact.
- $[0, \infty)$ with usual metric is not compact.
- Let M be an infinite set with discrete metric. Then M is not compact.
- Closed interval $[a, b]$ with usual metric is compact.
- Let M be a metric space. Let $A \subseteq M$. A is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\cup G_\alpha \supseteq A$ there exists a subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = A$.
- Any compact subset A of a metric space M is bounded.
- Any compact subset A of a metric space(M, d) is closed.
- A closed subspace of a compact metric space is compact.
- **Heine Borel Theorem:** Any closed interval $[a, b]$ is a compact subset of R.

NOTES

- A subset A of R is compact iff A is closed and bounded.
- A family $\mathcal{T}$ of subsets of a set M is said to have the finite intersection property if any finite members of $\mathcal{T}$ have non-empty intersection.
- A metric space M is compact iff any family of closed sets with finite intersection property has non-empty intersection.
- A metric space M is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite number of elements $x_1, x_2, \dots, x_n \in M$
  such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \dots . \cup B(x_n, \varepsilon) = M$.
- Any compact metric space is totally bounded.
- Let A be a subset of a metric space M. If A is totally bounded then A is bounded.
- Let (x_n) be sequence in a metric space M. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be an increasing sequence of positive integers. Then (x_{n_k}) is called a **subsequence** of (x_n).
- A metric space (M, d) is totally bounded iff every sequence in M has a Cauchy subsequence.
- A metric space M is said to be **sequentially compact** if every sequence in M has a convergent sun-sequence.

13.6 KEYWORDS

- **Open cover:** Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an ***open cover*** for M if $\cup G_\alpha = M$.
- **Subcover:** A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a ***subcover.***
- **Compact metric space:** A metric space M is said to be ***compact*** if every open cover for M has finite subcover.
- **Heine Borel Theorem:** Any closed interval $[a, b]$ is a compact subset of R.
- **Sequentially compact:** A metric space M is said to be **sequentially compact** if every sequence in M has a convergent sun-sequence.

13.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

1. If A and B are two compact subsets of a metric space M. Prove that $A \cup B$ is also compact.
2. Let M be a complete metric space. Prove that a closed subset A of M is compact if and only if A is totally bounded.
3. Prove that any Cauchy sequence in a metric space is totally bounded.

NOTES

4. Prove that any continuous function from a compact metric space to any other metric space is a closed map.

5. Any sequence in a compact metric space has a convergent subsequence.

6. Any continuous function defined on a closed interval $[a, b]$ is uniformly continuous.

13.8 FURTHER READINGS

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NOTES

UNIT-14 SEQUENCE OF FUNCTION AND SERIES OF FUNCTIONS

Structure

14.0 INTRODUCTION

In this chapter we discussed convergence of sequence and series of real numbers. In this chapter we discuss the convergence of sequence and series of functions. We deal almost exclusively with real-valued functions.

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what is meant by Pointwise convergence of sequence of functions
- Discuss Uniform Convergence of Sequence of functions
- Discuss Cauchy Criterion for Uniform Convergence

14.2 POINTWISE CONVERGENCE OF SEQUENCE OF FUNCTIONS

14.2.1 Definitions

Definition 1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. We say that $\{f_n\}_{n=1}^{\infty}$ converge to the function f on E if

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E)\ldots\ldots\ldots\ldots\ldots(1)$$

If (1) holds we sometimes say that $\{f_n\}_{n=1}^{\infty}$ converge *pointwise* to f on E. For if (1) holds, then, for every *point* x of E, the sequence

$\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges to $f(x)$. Here are several

examples.

If

$$f_n(x) = x^n \qquad (0 \le x \le 1),$$

then$\{f_n\}_{n=1}^{\infty}$ converges to f on [0,1] where

$$f(x) = 0 \qquad (0 \le x \le 1)$$

$$f(1) = 1.$$

For a second example let

$$g_n(x) = \frac{x}{1+nx} \qquad (0 \le x < \infty).$$

If $x > 0$, then $0 < g_n(x) \le \frac{x}{nx} = 1/n$. Hence

$$\lim_{n\to\infty} g_n(x) = 0 \qquad (x > 0).$$

Also, since $g_n(0) = 0$ for each $n \in I$, it is clear that $\{g_n(x)\}_{n=1}^{\infty}$

converge to 0 (the function identically 0) on $[0, \infty)$.

$$h_n(x) = \frac{nx}{1+n^2x^2} \qquad (-\infty < x < \infty).$$

Then if $x > 0$ we have

$$h_n(x) = \frac{1/nx}{\left(\frac{1}{n^2x^2}\right) + 1}$$

And hence $\lim_{n\to\infty} h_n(x) = 0$. Since $h_n(0) = 0$ for each $n \in I$ we see that $\{h_n(x)\}_{n=1}^{\infty}$.

For a fourth example let x_n denote the characteristic function of $[-n, n]$. For any $x \in R^1$ we have $\chi_n(x) = \chi_{n+1}(x) = \chi_{n+2}(x) = \cdots = 1$ provided $n \ge |x|$. (For then $x \in [-n, n]$). Hence

$$\lim_{n\to\infty} \chi_n(x) = 1 \qquad (x \in R^1),$$

And so $\{\chi_n\}_{n=1}^{\infty}$ converges to 1 on $(-\infty, \infty)$.

Definition 2. According to definition 1, the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ converges to f on the set E if, for each $x \in E$, given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N). \quad \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

In general, the number N depends on both ε and x. It is not always possible to find an N such that (1) holds for all x in E simultaniously.

For example, if $f_n(x) = x^n$ $(0 \leq x \leq 1)$, then, as we have seen, $\{f_n(x)\}_{n=1}^{\infty}$ converges to f on [0,1] where $f(x) = 0$ $(0 \leq x < 1)$ and $f(1) = 1$. With $\varepsilon = \frac{1}{2}$, then, for each $x \in E$, there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \frac{1}{2} \qquad (n \geq N). \quad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

If $x = 0$ or $x = 1$, then (2) is true for $N = 1$. However, if $x = \frac{3}{4} = 0.75$, then the smallest value of N for which (2) is true is $N = 3$. For, if $x = \frac{3}{4}$, then $f_n(x) = \left(\frac{3}{4}\right)^n$ while $f(x) = 0$. Thus, $|f_n(x) - f(x)| = \left(\frac{3}{4}\right)^n$, and $\left(\frac{3}{4}\right)^n < \frac{1}{2}$ if and only if $n \geq 3$.

Similarly, if $x = 0.9$ then the smallest value of N for which (2) is true is $N = 7$.

Indeed, there is no $N \in I$ such that (2) holds simultaneously for all $x \in [0,1]$.

For, if such an N existed, we would have

$$x^n < \frac{1}{2} \qquad (n \geq N)$$

For all x in [0,1]. This implies $x^n < \frac{1}{2}$ $(0 \leq x < 1)$. Letting $x \to 1^-$ we obtain the contradiction $1 \leq \frac{1}{2}$.

For the second example, the story is different. For, if

$$g_n(x) = \frac{x}{1+nx} \qquad (0 \leq x < \infty)\,,$$

Then $0 \leq g(x) \leq 1/n$ $\quad (1 \leq x < \infty)$. Hence, for any $\varepsilon > 0$ the statement

$$|g_n(x) - 0| < \varepsilon \qquad (n \geq N) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Is true for all x in $[0,\infty)$ simultaneously, provided only that $N > 1/\varepsilon$. (For in this case $|g_n(x) - 0| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ for all x in $[0,\infty)$.) Thus for this sequence $\{g_n(x)\}_{n=1}^{\infty}$ an $N \in I$ can be found such that (3) holds for all $x \in I$. This N depends only on ε and not on x.

Now consider

$$h_n(x) = \frac{nx}{1+n^2x^2} \qquad (-\infty < x < \infty).$$

We have seen that $\{h_n(x)\}_{n=1}^{\infty}$ converges to 0 on $(-\infty,\infty)$. Given $\varepsilon > 0$, we know therefore that for each $x \in (-\infty,\infty)$ there exists $N \in I$ such that

$$|h_n(x) - 0| < \varepsilon \qquad (n \geq N). \quad \text{....................(4)}$$

However, note that $h_n\left(\frac{1}{n}\right) = \frac{1}{2}$. Hence, if $\varepsilon = \frac{1}{2}$, there is no single $N \in I$ such that (4) holds simultaneously for all $x \in (-\infty,\infty)$. For if such an N existed we would have

$$h_N(x) < \frac{1}{2} \qquad (-\infty < x < \infty).$$

But if $x = 1/N$ this a contradiction.

We leave it to the reader to show that if $\varepsilon < 1$ then there is no $N \in I$ such that the statement

$$|x_n(x) - 1| < \varepsilon \qquad (n \geq N)$$

Holds for all real x simultaneously, where χ_n is as in the fourth example.

14.3 UNIFORM CONVERGENCE OF SEQUENCE OF FUNCTIONS

We have agreed to say that $\{f_n(x)\}_{n=1}^{\infty}$ converges (pointwise) to f on E if, each $x \in E$, given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N). \quad \text{..................(1)}$$

We have seen several examples in which in which it is impossible to find an N such that (1) holds for all $x \in E$ simultaneously.

If for each $\varepsilon > 0$ it is possible to find an N such that (1) holds for all $x \in E$ then we say that $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to f on E.

Definition 3. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued function on a set E. We say that $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to the function f on E if given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N; x \in E).$$

The wording of this definition implies that N depends on ε but not on x. It is clear that if $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to f on E, then $\{f_n(x)\}_{n=1}^{\infty}$ converges poitwise to f on E.

Thus, if $g_n(x) = \frac{x}{(1+nx)}$ $(0 \leq x < \infty)$, then our work in previous section shows that $\{g_n(x)\}_{n=1}^{\infty}$ converges uniformly to 0 on $[0, \infty)$. For we have already shown that given $\varepsilon > 0$ there exists $N \in I$ such that

$$|g_n(x) - 0| < \varepsilon \qquad (n \geq N; 0 \leq x < \infty).$$

(Any N such that $N > 1/\varepsilon$ will do).

It is not too easy to state what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ not to converge uniformly to f on E. We shall now do this.

Corollary 1. The sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f on E if and only if there exists some $\varepsilon > 0$ such that there is no $N \in I$ for which the statement

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N; x \in E)$$

holds.

The reader should not proceed until he is convinced that this section is equivalent to the previous section.

If $f_n(x) = x^n$ $(0 \leq x \leq 1)$ and $f(x) = 0$ $(0 \leq x \leq 1)$, $f(1) = 1$, then we have seen that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on $[0,1]$. However, $\{f_n\}_{n=1}^{\infty}$ dose not converge uniformly to f on $[0,1]$. For, as we saw in Definition 2, if $\varepsilon = \frac{1}{2}$ then there is no $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N; 0 \leq x \leq 1).$$

If E is an interval of real numbers then it is readily seen that saying $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on E means that given $\varepsilon > 0$ there exists $N \in I$ such that the vertical distance from any point on the graph of f to the corresponding point on the graph of any of the functions $f_N, f_{N+1}, \dots$ is less than ε. Thus, if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on E, then the graphs of $f_N, f_{N+1}, \dots$ are all "uniformly close" to the graph of f.

In particularly, if $\{f_n\}_{n=1}^{\infty}$ converges to zero uniformly on E, then given $\varepsilon > 0$ there exists $N \in I$ such that the graphs of $f_N, f_{N+1}, \dots$ are all within vertical distance ε of the $x - axis$.

Here is still another way to view uniform convergence.

Definition 4. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to 0on E, then given $\varepsilon > 0$ there exists N such that

$$|f_n(x)| < \varepsilon \qquad (n \geq N; x \in E).$$

This implies

$$\underset{x \in E}{l.u.b.} |f_n(x)| \leq \varepsilon \qquad (n \geq N).$$

Hence, if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to zero on E, then

$$\lim_{n \to \infty} \underset{x \in E}{l.u.b.} |f_n(x)| = 0. \quad \dots\dots\dots\dots\dots(1)$$

Conversely, it not difficult to show that if (1) holds then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to 0 on E.

This readily proves that the sequence $\{h_n\}_{n=1}^{\infty}$ of section 14.1 does not converge uniformly to zero on $(-\infty, \infty)$. For

$$\underset{-\infty < z < \infty}{l.u.b.} |h_n(x)| \geq \left| h_n\left(\frac{1}{n}\right) \right| = \frac{1}{2} \qquad (n = 1,2, \dots),$$

and hence $\underset{-\infty < z < \infty}{l.u.b.} |h_n(x)|$ cannot approach zero as $n \to \infty$.

Definition 5. From Definition 3 it follows immediately that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on E if and only if $\{f_n - f\}_{n=1}^{\infty}$ converges uniformly to 0 on E. From Definition 4 we then have

Theorem 1. The sequence of function $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on E if and only if

$$\underset{x \in E}{l.u.b.} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

14.3.1 Cauchy Criterion for Uniform Convergence

In this subsection, the Cauchy criterion for uniform convergence. It is analogues to the result that a sequence of real numbers is convergent If and only if it is Cauchy.

Theorem 2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. Then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on E (to some function f) if and only if given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (m, n \geq N; x \in E). \ldots\ldots\ldots\ldots(1)$$

Proof. Suppose first that $\{f_n\}_{n=1}^{\infty}$ is a uniformly convergent sequence of functions on E, converging to f on E. Then, given $\varepsilon > 0$, there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \qquad (n \geq N; x \in E).$$

Thus, if $m, n \geq N$ we have, for any $x \in E$,

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

and hence (1) holds for this N.

Conversely let $\{f_n\}_{n=1}^{\infty}$ be any sequence of functions on E such that, given $\varepsilon > 0$, there exists $N \in I$ such that (1) holds. We must show that there is a function f on E such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on E. From (1) we see that, for each fixed $x \in E$, the sequence of real numbers $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence $m_{n\to\infty} f_n(x)$ exists for each $x \in E$. Define f by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E).$$

Keeping m fixed in (1) and letting $n \to \infty$ we obtain

$$|f_m(x) - f(x)| \leq \varepsilon \qquad (m \geq N; x \in E).$$

Since ε was arbitrary, this shows that $\{f_m\}_{m=1}^{\infty}$ converges uniformly to f on E, and the proof is complete.

The next results, called Dini's theorem, shows that under a very special set of circumstances a sequence of *continuous* functions must converge uniformly.

Theorem 3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on the *compact* metric space $\langle M, \rho \rangle$ such that

$$f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots \qquad (x \in M). \quad \ldots\ldots\ldots\ldots(1)$$

If $\{f_n\}_{n=1}^{\infty}$ converges (*pointwise*) on M to the continuous function f, then $\{f_n\}_{n=1}^{\infty}$ converges *uniformly* to f on M.

Proof. For each $n \in I$ let $g_n = f - f_n$. Then from (1) we have

$$g_1(x) \geq g_2(x) \geq \cdots \geq g_n(x) \geq \cdots \qquad (x \in M). \quad \ldots\ldots\ldots\ldots(2)$$

Also, since $\{f_n\}_{n=1}^{\infty}$ converges to f on M we have

$$\lim_{n\to\infty} g_n(x) = 0 \qquad (x \in M). \quad \ldots\ldots\ldots\ldots(3)$$

We must show that $\{g_n\}_{n=1}^{\infty}$ converges uniformly to 0 on M.

Fix $\varepsilon > 0$. If $x \in M$, then (3) assures us of the existence of $N(x) \in I$ such that

$$g_{N(x)}(x) < \frac{\varepsilon}{2}.$$

Since $g_{N(x)}$ is continuous at x, there is an open ball B_x about x such that

$$g_{N(x)}(x) < \varepsilon \qquad (y \in B_x).$$

The B_x for all $x \in M$ from an open covering of M. Since this a finite number of the B_x—say

$$B_{x_1}, B_{x_2}, \ldots, B_{x_k}$$

also cover M. Let $N = \max[N(x_1), \ldots, N(x_k)]$. Now if y is any point in M, then $y \in B_{x_j}$ for some $j = 1, \ldots, k$. Hence

$$g_{N(x_j)}(y) < \varepsilon.$$

But since $N(x_j) \leq N$, (2) implies

$$g_N(y) \leq g_{N(x_j)}(y).$$

Hence

$$0 \leq g_N(y) < \varepsilon$$

For all $y \in M$. But then (2) shows that

$$0 \leq g_n(y) < \varepsilon \qquad (n \geq N; y \in M),$$

And so $\{g_n\}_{n=1}^{\infty}$ converges uniformly to 0 on M. This completes the proof.

It is clear that Theorem 3 remains true if the inequality signs in (1) are all reversed. For then we could set $g_n = f - f_n$ and proceed as above.

Check your progress

1. Define converge *pointwise*.
2. Define convergent sequence of functions.

14.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. We say that $\{f_n\}_{n=1}^{\infty}$ converge to the function f on E if

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E)$$

If (1) holds we sometimes say that $\{f_n\}_{n=1}^{\infty}$ converge *pointwise* to f on E. For if (1) holds, then, for every *point* x of E, the sequence

$\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges to $f(x)$.

2. The sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ converges to f on the set E if, for each $x \in E$, given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N).$$

In general, the number N depends on both ε and x.

14.5 SUMMARY

- The sequence of function $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** to f on E if and only if

$$\underset{x\in E}{l.u.b.}|f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

- Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. We say that $\{f_n\}_{n=1}^{\infty}$ converge to the function f on E if

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E)$$

If (1) holds we sometimes say that $\{f_n\}_{n=1}^{\infty}$ **converge *pointwise*** to f on E. For if (1) holds, then, for every *point* x of E, the sequence

- The **sequence of functions** $\{f_n(x)\}_{n=1}^{\infty}$ **converges** to f on the set E if, for each $x \in E$, given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N).$$

In general, the number N depends on both ε and x.

- Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. Then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on E (to some function f) if and only if given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (m, n \geq N; x \in E).$$

- Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on the *compact* metric space $\langle M, \rho \rangle$ such that

$$f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots \qquad (x \in M).$$

14.6 KEYWORDS

Converges uniformly: Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued function on a set E. We say that $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to the function f on E if given $\varepsilon > 0$ there exists $N \in I$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N; x \in E).$$

14.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

1) If $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ converges uniformly on E, prove that $\{f_n + g_n\}_{n=1}^{\infty}$ converges uniformly on E.
2) Let A be a dense subset of the metric space M. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on M, and if $\{f_n\}_{n=1}^{\infty}$ converges uniformly on A, prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on M.
3) The uniform limit of a sequence of discontinuous functions can be continuous.
4) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions which converges uniformly to the continuous function f on $(-\infty, \infty)$, prove that $\lim_{n\to\infty} f_n\left(x + \frac{1}{n}\right) = f(x) \qquad (-\infty < x < \infty)$.
5) If $(f_n) \to f$ and each f_n and f are continuous then the convergence is uniform.

14.8 FURTHER READINGS

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